Bodily Tides

near

Spin-Orbit Resonances

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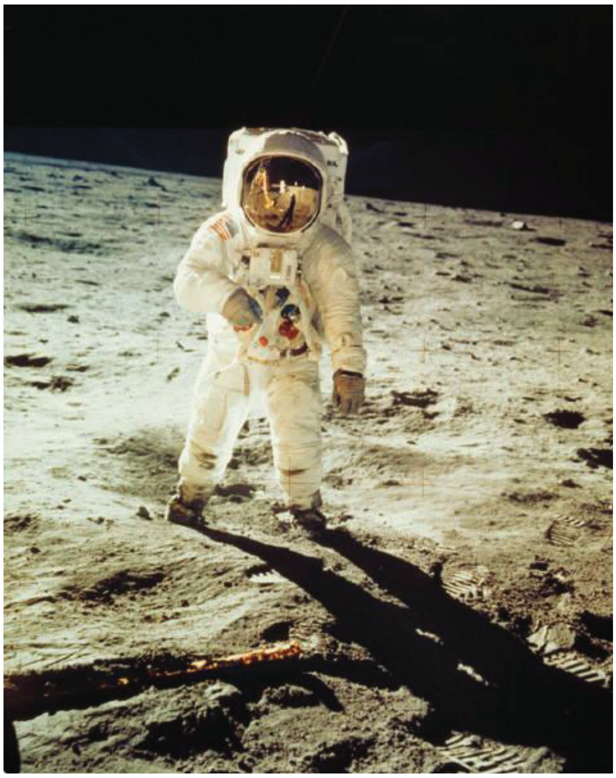
Why tides matter



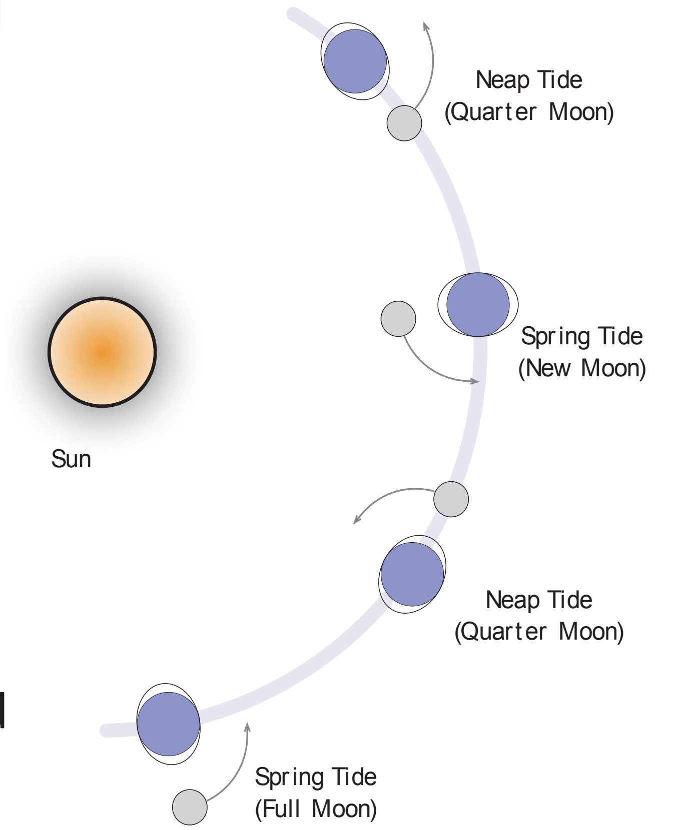
- The Moon is facing Earth with one side
- Mercury makes 3 sidereal rotations over 2 orbital revolutions around the Sun (discovered in 1965)
- Tides play a crucial role in the dynamics of exoplanets (spin states, circularisation of orbits, tidal heating). Thereby tides influence greatly these planets' chances of being habitable.

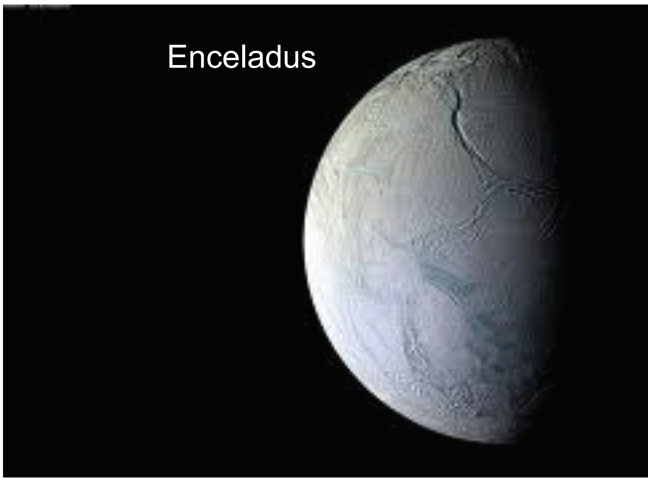
Ocean tides and bodily tides

Usually the concept of tides is associated with rising and webbing sea level due to the gravitational pull from the Moon and the Sun (ocean tides)

It is harder to notice that the solid earth also bulks in response to this pull.

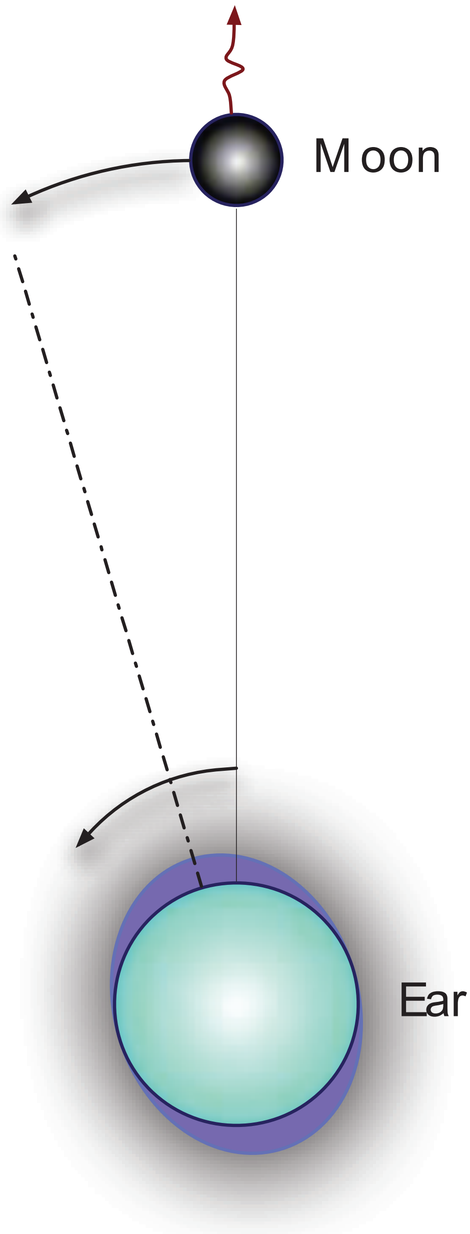
The interior of Enceladus is constantly heated up, most likely by bodily tides, resulting in cryovolcanism and frequent resurfacing





The Moon is drifting away

- The gravitational pull of the Moon generates a two-sided, symmetric bulge
- The reaction is not instantaneous, but slightly delayed (*tidal lagging*)
- As the Earth is rotating faster than the Moon is orbiting it, the bulge is running *ahead* of the line of centres
- The leading bulge generates an orbital torque in the same direction as the orbital motion of the Moon
- As a result, the Moon accelerates and drifts away from the Earth (3.8 cm per year)

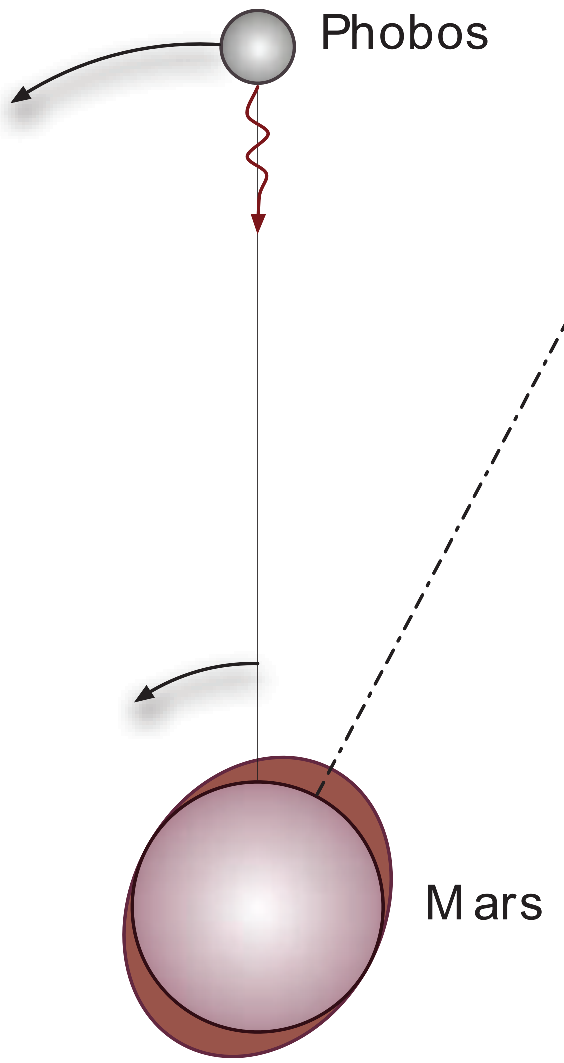


Phobos is falling down

- Mars is rotating slower than Phobos is orbiting it. Therefore, the tidal bulge is lagging *behind* the line of centres connecting the two bodies.

- The lagging bulge generates an orbital torque in the opposite direction to the orbital motion of Phobos.

- As a result, Phobos is slowing down and is falling onto Mars (~100 m per year)



The plethora of exoplanets

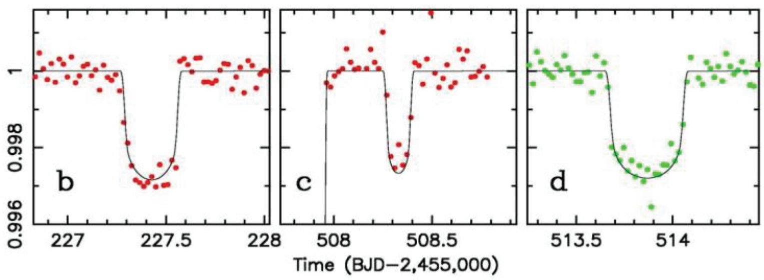
- 840+ exoplanets detected, 2300+ candidates from *Kepler* awaiting confirmation

- Diversity of types: terrestrials, gas giants, ice giants, water worlds

- The frequency of terrestrial planets in the habitable zone around FGK stars ~34 ± 14% (Traub 2012) – we may have 10^{10} habitable planets in the Milky Way alone

- Orbital period distribution follows a power law, $dN/dP \sim P^{-0.3}$, but it drops abruptly at P = 3 days – stars devour closer planets, which is a tidal “Mars+Phobos” effect

- Planets’ spin is as important for habitability as stellar irradiation or orbital eccentricity – consider Mercurian day which is 176 Earth’s solar days



Transits of exoplanets around Kepler 34 and 35, from Welsh et al 2012, Nature 481

WHAT WILL BE COVERED IN THIS LECTURE:

THE WAY TIDES INFLUENCE THE SPIN OF CELESTIAL BODIES.

SPECIFICALLY: FREQUENCY-DEPENDENCE OF TIDAL TORQUES.

Among other things, the shape of this dependence defines the probabilities of capture into spin-orbit resonances.

These probabilities have ramifications for habitability:

you won't feel comfortable on a planet, which is close to its star and is always showing to the same side to the star.

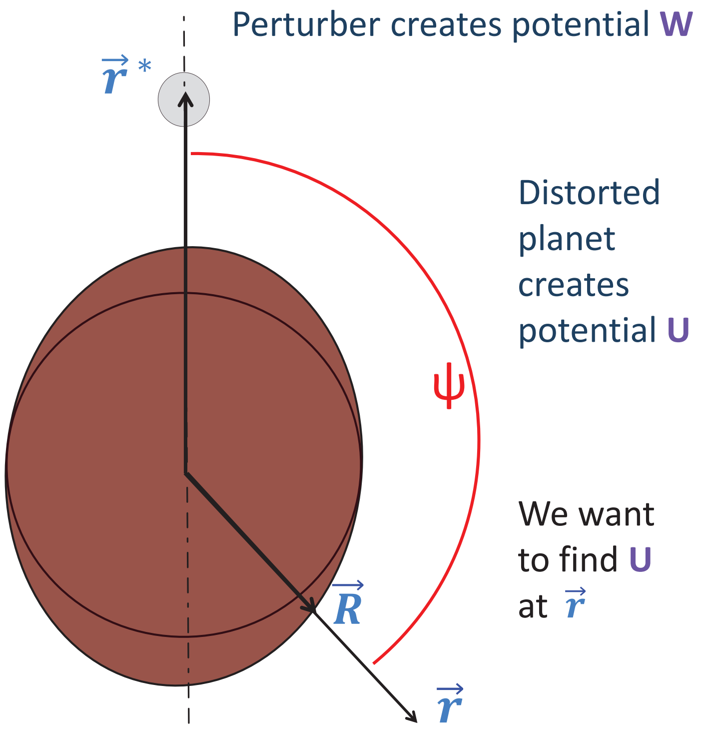
A faster-spinning planet would have a more balanced climate.

WHAT WILL NOT BE COVERED:

TIDAL EVOLUTION OF ORBITS.

TIDAL HEATING.

STATIC TIDE



TWO-STEP METHOD TO FIND U AT POINT $\vec{r}$:

STEP 1. Take a point $\vec{R}$ on the planet's surface, right beneath $\vec{r}$.

Calculate the potential W created in $\vec{R}$ by the perturber:

$$W(\vec{R},\,\vec{r}^{\,*}) \;=\; \sum_{l=2}^{\infty} W_l(\vec{R},\,\vec{r}^{\,*}) \quad \text{where} \quad W_l(\vec{R},\,\vec{r}^{\,*}) \,\propto\, P_l(\cos\psi)$$

STEP 2. Knowing the perturbing potential W in point $\vec{R}$, calculate

the tidal-response potential U in the point $\vec{r}$ above $\vec{R}$:

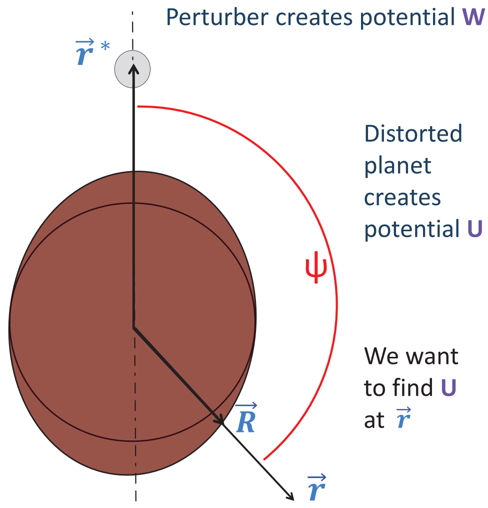
$$U(\vec{r}) \;=\; \sum_{l=2}^{\infty} U_l(\vec{r}) \quad \text{where} \quad U_l(\vec{r}) \;=\; k_l \left(\frac{R}{r}\right)^{l+1} W_l(\vec{R},\,\vec{r}^{\,*})$$

Sphere of density ρ, radius R, rigidity μ : the principal *static* Love number is

$$k_2 \;=\; \frac{3}{2}\,\frac{1}{1+\frac{19}{2}\,\frac{\mu}{\mathrm{g}\,\rho\,R}} \;=\; \frac{3}{2}\,\frac{1}{1+\frac{57}{8\,\pi}\,\frac{1}{G\,\rho^2\,R^2\,J}} \quad .$$

where we switched to the compliance $J \equiv 1/\mu$.

Similar formulae for all degrees $l > 2$.



WHAT CAN WE GET FROM THESE FORMULAE?

$$W(\vec{\boldsymbol{R}},\,\vec{\boldsymbol{r}}^{\,*}) \;=\; \sum_{l=2}^{\infty} W_l(\vec{\boldsymbol{R}},\,\vec{\boldsymbol{r}}^{\,*}) \;\;,\;\; \text{where} \quad W_l(\vec{\boldsymbol{R}},\,\vec{\boldsymbol{r}}^{\,*}) \;=\; -\,\frac{G\,M_{moon}}{r^{\,*}}\left(\frac{R}{r^{\,*}}\right)^{l} P_l(\cos\boldsymbol{\psi})\;\;,$$

$$U(\vec{\boldsymbol{r}}) \;=\; \sum_{l=2}^{\infty} U_l(\vec{\boldsymbol{r}}) \quad , \qquad \text{where} \quad U_l(\vec{\boldsymbol{r}}) \;=\; k_l\left(\frac{R}{r}\right)^{l+1} W_l(\vec{\boldsymbol{R}},\,\vec{\boldsymbol{r}}^{\,*}) \quad .$$

WE CAN EXPRESS **U** VIA ORBITAL ELEMENTS OF $\vec{\boldsymbol{r}}^{\,*}$ AND $\vec{\boldsymbol{r}}$.

Insertion of W_l in U_l gives :
$$U_l(\vec{\boldsymbol{r}}) \;=\; -\,G\,M_{moon}\;k_l\;\frac{R^{2l+1}}{r^{\,l+1}\,r^{\,*\,l+1}}\;P_l(\cos\psi) \quad .$$

Express $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{r}}^{\,*}$ through the Kepler elements:

$\vec{\boldsymbol{r}}^{*} = (\,a^*,\,e^*,\,i^*,\,\Omega^*,\,\omega^*,\,\mathcal{M}^*\,)$ and $\vec{\boldsymbol{r}} = (\,a,\,e,\,i,\,\Omega,\,\omega,\,\mathcal{M}\,)$.

Arrive at

$$U(\vec{\boldsymbol{r}}) = -GM_{moon}\sum_{l=2}^{\infty}\sum_{m=0}^{l}\sum_{p=0}^{\infty}\sum_{q=-\infty}^{\infty} k_l \times \left\{\begin{matrix}\text{function of integers } lmpq \text{ and the}\\ \text{Kepler elements of both } \vec{\boldsymbol{r}}^{\,*}\text{and } \vec{\boldsymbol{r}}\end{matrix}\right\}$$

Darwin (1879) derived a partial sum,[1] Kaula (1964) wrote down the full series.

How does the perturber perturb itself by the tides it creates in planet?

Set $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}^{\,*}$, obtain $U(\vec{\boldsymbol{r}}^{\,*})$. The resulting force will be central,

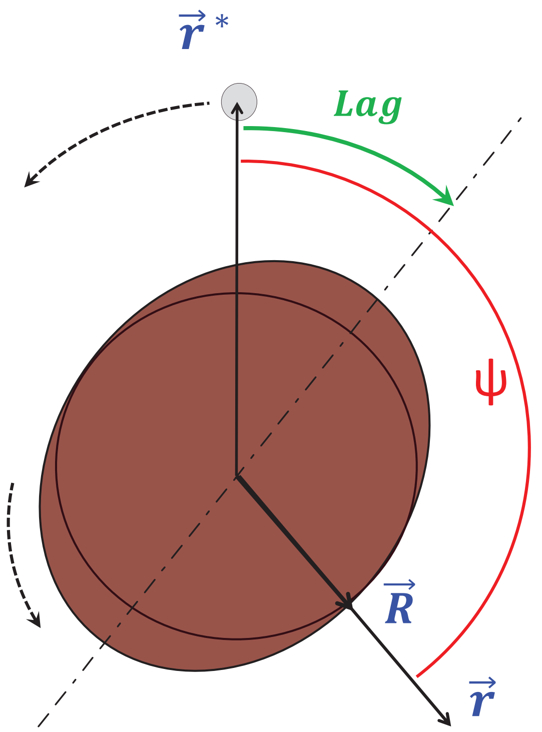
and the torque will be nil — which is natural for a static tide.

[1] For introduction into Darwin's method see Ferraz-Mello, Rodríguez, & Hussmann (2008).

DYNAMICAL TIDE

When the orbital angular velocity of the perturber is faster than the planetary spin, the bulge falls behind.

(The case of Phobos)



The same problem, though now in motion:

The perturber produces potential $\mathbf{W}$ which now is time-dependent.

A wave of deformation circulates over the planet.

(In fact, many waves of various periods, as it will turn out.)

The so-deformed planet generates a time-dependent tidal potential $\mathbf{U}$.

We want to find $\mathbf{U}$ in an exterior point $\vec{\boldsymbol{r}}$.

(Of a special interest is the value of $\mathbf{U}$ in the point $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}^{\,*}$ where the perturber is located. It will render us the tidal torque acting on the perturber and the opposite torque acting on the planet.)

Have to extend Love's solution to dynamical tides.

How exactly did Love and Darwin treat the static case?

STEP 1 is easy. In an arbitrary point $\vec{\boldsymbol{R}}$ on the planet's surface, expand the perturbing potential $\mathbf{W}$ over the Legendre polynomials:

$$W(\vec{\boldsymbol{R}},\, \vec{\boldsymbol{r}}^{\,*}) \;=\; \sum_{l=2}^{\infty} W_l(\vec{\boldsymbol{R}},\, \vec{\boldsymbol{r}}^{\,*}) \quad ,$$

where

$$W_l(\vec{\boldsymbol{R}},\, \vec{\boldsymbol{r}}^{\,*}) \;=\; -\;\frac{G\,M_{moon}}{r^{\,*}}\left(\frac{R}{r^{\,*}}\right)^{l} P_l(\cos\boldsymbol{\psi}) \quad , \qquad \boldsymbol{\psi} \;=\; \widehat{(\vec{\boldsymbol{R}}\,,\, \vec{\boldsymbol{r}}^{\,*})}$$

STEP 2 Combine the constituency equation with the Second Law of Newton:

$$2\;u_{\beta\nu} \;=\; J\;\sigma_{\beta\nu} \quad , \qquad \text{/linear, isotropic material/} \qquad (1)$$

$$0 \;=\; \frac{\partial\sigma_{\beta\nu}}{\partial x_{\nu}} \;-\; \rho\;\frac{\partial(W\;+\;U)}{\partial x_{\beta}} \quad , \qquad (2)$$

where $W = \sum W_l$ and $U = \sum U_l$,

$\sigma_{\beta\nu}$, $u_{\beta\nu}$ are the stress and strain tensors, ρ is density, J is compliance.

(Compliance is inverse to rigidity $J \equiv 1/\mu$.)

With boundary conditions, eqns (1) and (2) yield: $U_l(\vec{\boldsymbol{r}}) = k_l\left(\frac{R}{r}\right)^{l+1} W_l(\vec{\boldsymbol{R}})$,

where $\vec{\boldsymbol{r}}$ is located right above $\vec{\boldsymbol{R}}$, and the Love numbers are

$$k_2 \;=\; \frac{3}{2}\;\frac{1}{1+\frac{19}{2}\;\frac{\mu}{\mathrm{g}\,\rho\,R}} \;=\; \frac{3}{2}\;\frac{1}{1+\frac{57}{8\,\pi}\;\frac{1}{G\,\rho^2\,R^2\,J}} \quad , \qquad J \equiv 1/\mu \quad .$$

Similar formulae for k_l with $l > 2$.

What is different in dynamics?

(a) In (1), strain will lag behind stress, due to friction.

(b) In (2), we shall get acceleration and inertial forces.

STATICS: $$2\, u_{\beta\nu} \;=\; J\, \sigma_{\beta\nu} \quad , \tag{1}$$

$$0 \;=\; \frac{\partial \sigma_{\beta\nu}}{\partial x_\nu} \;-\; \rho\, \frac{\partial (W\; +\; U)}{\partial x_\beta} \quad . \tag{2}$$

DYNAMICS: $$2u_{\gamma\nu}(t) = \int^{t} \dot{J}\,(t - t')\, \sigma_{\gamma\nu}(t') dt' \quad , \tag{1$'$}$$

$$\rho\, \ddot{u}_\beta \;=\; \frac{\partial \sigma_{\beta\nu}}{\partial x_\nu} \;-\; \rho\, \frac{\partial (W\; +\; U)}{\partial x_\beta} \;+\; \text{inert. forces} \tag{2$'$}$$

SWITCH TO FOURIER IMAGES: $\sigma_{\gamma\nu}(t) \;=\; \int_{-\infty}^{\infty} \bar{\sigma}_{\gamma\nu}(\omega)\, e^{\,i\omega t}\, d\omega$, etc.

NEGLECT ACCELERATIONS AND INERTIAL FORCES:

$$2\; \bar{u}_{\beta\nu}(\omega) \;=\; \bar{J}(\omega)\; \bar{\sigma}_{\beta\nu}(\omega) \quad , \tag{1$''$}$$

$$0 \;=\; \frac{\partial \bar{\sigma}_{\beta\nu}(\omega)}{\partial x_\nu} \;-\; \rho\; \frac{\partial \left[\bar{W}(\omega) \;+\; \bar{U}(\omega)\right]}{\partial x_\beta} \quad , \tag{2$''$}$$

Eqns (1$''$ - 2$''$) mimic (1 - 2). The solutions, too, should mimic one another:

$$\bar{U}_l(\omega) \;=\; \bar{k}_l(\omega)\; \left(\frac{R}{r}\right)^{l+1}\; \bar{W}_l(\omega)\,, \tag{3}$$

where the complex Love numbers are

$$\bar{k}_2(\omega) \;=\; \frac{3}{2}\; \frac{1}{1 + \frac{19}{2}\; \frac{\bar{\mu}(\omega)}{\mathrm{g}\, \rho\, R}} \;=\; \frac{3}{2}\; \frac{1}{1 + \frac{57}{8\,\pi}\; \frac{1}{G\, \rho^2\, R^2\, \bar{J}(\omega)}} \quad , \qquad \bar{J}(\omega) \equiv 1/\bar{\mu}(\omega)$$

and similar formulae for all $\bar{k}_l(\omega)$, $\quad l > 2$.

FOUR CONCLUSIONS:

1. At each Fourier mode ω, the complex amplitudes $\bar{U}(\omega)$ and $\bar{W}(\omega)$ relate in the same algebraic manner as U and W in the static problem:

$$\bar{U}_l(\omega) \;=\; \bar{k}_l(\omega)\,\left(\frac{R}{r}\right)^{l+1}\,\bar{W}_l(\omega) \qquad\qquad (\circledast)$$

where $\bar{k}_l(\omega)$ are expressed via $\bar{J}(\omega)$ in the same algebraic way as k_l were expressed through $J = 1/\mu$ in statics:

$$\bar{k}_l(\omega) = \frac{3}{2(l-1)}\;\frac{1}{1\,+\,\dfrac{3(2l^2+4l+3)}{4l\pi G\rho^2 R^2\,\bar{J}(\omega)}}$$

2. For absolute values, $(\circledast)$ yields: $\;|\,\bar{U}_l(\omega)\,| \;=\; |\,\bar{k}_l(\omega)\,|\,\left(\frac{R}{r}\right)^{l+1}\,|\,\bar{W}_l(\omega)\,|\;$,

so the absolute value $|\,\bar{k}_l(\omega)\,|$ plays the role of dynamical Love number.

3. Let ϵ_l be the negative phase of the complex Love number:

$$\bar{k}_l(\omega) = |\,\bar{k}_l(\omega)\,|\,\exp\left[\,-\,i\,\epsilon_l(\omega)\,\right]\;.$$

Then, for the phases of $\bar{U}(\omega)$ and $\bar{W}(\omega)$ at each mode ω, eqn $(\circledast)$ yields:

$$\phi_U(\omega) \;=\; \phi_W(\omega) - \epsilon_l(\omega)$$

Thus, the phase $\epsilon_l(\omega)$ of the Love number plays the role of the tidal lag.

4. Naturally, $(\circledast)$ also yields $\;U_l(\vec{\boldsymbol{r}},\,t) \;=\; \left(\frac{R}{r}\right)^{l+1}\int_{-\infty}^{t}\dot{k}_l(t-t\,')\;W_l(\vec{\boldsymbol{R}}\,,\;\vec{\boldsymbol{r}}^{\,*},\;t\,')\;dt\,'\;$,

which is similar to the constitutive equation $\;2\,u_{\beta\nu}(t) \;=\; \int_{-\infty}^{t}\dot{J}_l(t-t\,')\;\sigma_{\beta\nu}(t\,')\;dt\,'$ connecting the present-time strain $u_{\beta\nu}(t)$ with the stress $\sigma_{\beta\nu}(t\,')$ at $t\,'\leq t\,.$

Above we agreed that, at each Fourier mode ω, the complex amplitudes $\bar{U}(\omega)$ and $\bar{W}(\omega)$ relate in the same algebraic way as static U and W:

$$\bar{U}_l(\omega) \;=\; \bar{k}_l(\omega)\,\left(\frac{R}{r}\right)^{l+1}\;\bar{W}_l(\omega) \qquad\qquad (\divideontimes)$$

where $\bar{k}_l(\omega)$ are expressed via $\bar{J}(\omega)$ in the same algebraic way as k_l were expressed through $J = 1/\mu$ in statics:

$$\bar{k}_l(\omega) = \frac{3}{2(l-1)}\;\frac{1}{1\,+\,\dfrac{3(2l^2+4l+3)}{4l\pi G\rho^2 R^2\,\bar{J}(\omega)}} \qquad\qquad (\divideontimes\divideontimes)$$

WE NEED TO FIND THE NEGATIVE IMAGINARY PART OF THE COMPLEX LOVE NUMBER $\bar{k}_l(\omega)$, BECAUSE IT WILL SHOW UP IN A FORMULA FOR THE TIDAL TORQUE.

TURN - OF - THE - CRANK METHOD :

1. Grab any rheology $\bar{J}(\omega)$, plug it in $(\divideontimes\divideontimes)$, get $\bar{k}_l(\omega)$.

[Recall that $\bar{J}(\omega)$ is the Fourier transform of the kernel $\dot{J}_l(t-t\,')$ of the operator $2\,u_{\beta\nu}(t) = \int_{-\infty}^{t}\dot{J}_l(t-t\,')\,\sigma_{\beta\nu}(t\,')\,dt\,'$. So $\bar{J}(\omega)$ contains all info on rheology.]

2. From $\bar{k}_l(\omega)$, get its negative imag. part $k_l(\omega)\,\sin\epsilon_l(\omega) \;=\; -\,\mathcal{I}m\left[\,\bar{k}_l(\omega)\,\right]$.

STRATEGY FOR CALCULATION OF THE TIDAL TORQUE:

- TAKE $U(\vec{r})$ AT THE PERTURBER'S LOCATION $\vec{r} = \vec{r}^{\,*}$

RECALL WHAT WE HAD IN STATICS:

$$U(\vec{r}) = -GM_{moon} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} k_l \times \left\{ \begin{array}{l} \text{function of integers } lmpq \text{ and the} \\ \text{Kepler elements of both } \vec{r}^{\,*} \text{and } \vec{r} \end{array} \right\}$$

where k_l was the static Love number.

IN DYNAMICS, THIS SERIES BECOMES A FOURIER ONE:

$$U(\vec{r}) = -GM_{moon} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} k_l \cos\epsilon_l \times \left\{ \begin{array}{l} \text{function of integers } lmpq \text{ and the} \\ \text{Kepler elements of both } \vec{r}^{\,*} \text{and } \vec{r} \end{array} \right\}$$

An $lmpq$ term corresponds to Fourier mode $\omega_{lmpq} = (l-2p+q)n - m\,\dot{\theta}$

/ n and $\dot{\theta}$ being the mean motion and spin rate / ,

while $k_l \cos\epsilon_l = k_l(\omega_{lmpq}) \cos\epsilon_l(\omega_{lmpq}) = \mathcal{R}e\left[\bar{k}_l(\omega_{lmpq})\right]$

- FROM THAT, GET THE TORQUE ACTING ON PERTURBER
- AN EQUAL BUT OPPOSITE TORQUE IS ACTING ON PLANET

$$\mathcal{T} = \mathcal{T}^{secular} + \mathcal{T}^{oscillating}$$

$$\mathcal{T}^{secular} = -GM_{moon} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} k_l \sin\epsilon_l \times \left\{ \begin{array}{l} \text{function of integers } lmpq \text{ and the} \\ \text{Kepler elements of both } \vec{r}^{\,*} \text{and } \vec{r} \end{array} \right\}$$

where

$$k_l \sin\epsilon_l = k_l(\omega_{lmpq}) \sin\epsilon_l(\omega_{lmpq}) = -\mathcal{I}m\left[\bar{k}_l(\omega_{lmpq})\right]$$

This is why we were so eager to find the complex Love number $\bar{k}_l(\omega)$!!

Without a link to rheology, $\mathcal{T}^{secular}$ was written down by Goldreich (1966).

It is very common to misidentify k_l with the static Love number,
and equally common to misidentify $\sin\epsilon_l$ with the inverse seismic Q factor.

Goldreich (1963) warned against the latter, but his admonition went unnoticed.

To study tidal despinning of bodies, and to model their entrapment

into spin-orbit resonances, we need to know the tidal torque

$$\mathcal{T}^{\,secular} = \sum_{l=2}^{\infty}\sum_{m=0}^{l}\sum_{p=0}^{\infty}\sum_{q=-\infty}^{\infty} \ldots \; k_l(\omega_{lmpq}) \; \sin\epsilon_l(\omega_{lmpq}) \quad ,$$

Hence the need to know the functions $\;|\,\bar{k}_l\,(\omega)\,|\;\sin\epsilon_l\,(\omega) \;=\; -\,\mathcal{I}m\left[\,\bar{k}_l(\omega)\,\right]$,

where $\quad \bar{k}_l(\omega) = \dfrac{3}{2}\;\dfrac{1}{\mathbf{1}\,+\,\dfrac{3(2l^2+4l+3)}{4l\pi G\rho^2 R^2\;\bar{J}(\omega)}} \qquad$ and $\qquad \omega = \omega_{lmpq}$

COMPETITION: Self-gravitation versus Rheology, i.e.,

$$\mathbf{1} \qquad \text{versus} \qquad \frac{3(2l^2+4l+3)}{4l\pi G\rho^2 R^2\;\bar{J}(\omega)}$$

If not for this "**1**", then:

– our $\bar{k}_l(\omega)$ would mimic $\bar{J}(\omega)$,

– tidal lags ϵ_l would coincide with the seismic lag δ,

– all the tidal factors $\;\dfrac{k_l(\omega)}{Q_l(\omega)}\;\text{Sgn}(\omega) = |\,\bar{k}_l(\omega)\,|\,\sin\epsilon_l(\omega) = \;-\,\mathcal{I}m\left[\,\bar{k}_l(\omega)\,\right]$
would mimic their seismic counterpart $\;1/Q(\omega) = \sin\delta(\omega) \;=\; -\,\mathcal{I}m\left[\,\bar{J}(\omega)\,\right]$

So this "**1**" term

– reflects the (self-gravity-caused) difference between the tidal and seismic dissipation,

– guarantees that $k_l(\omega)/Q_l(\omega)$ smoothly goes through nil when ω does so (crossing of a resonance)

– excludes any possibility of discontinuity in $\bar{k}(\omega)$ and $k_l(\omega)/Q_l(\omega)$, no matter what rheology we choose.[2]

[2] From time to time, different authors claim that "bad rheologies" render discontinuities in $\dfrac{k_l(\omega)}{Q_l(\omega)}$ at $\omega\rightarrow 0$. Discontinuities stem not from "bad" rheologies but from bad math.

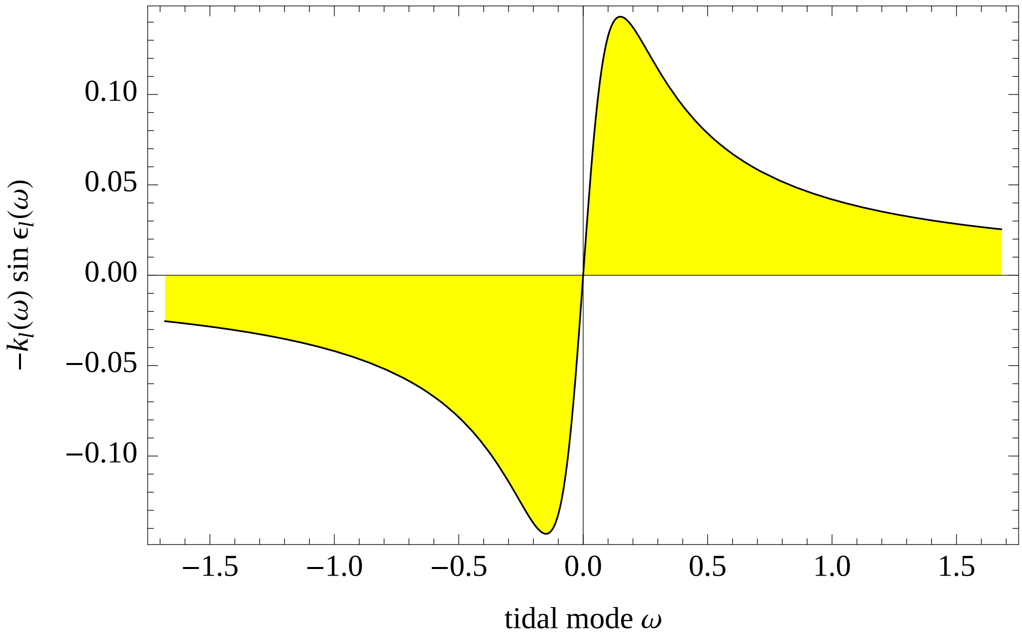
$\mid \bar{k}_l(\omega) \mid \sin \epsilon_l(\omega) = \; - \mathcal{I}m \left[\bar{k}_l(\omega) \right]$ is often denoted as $\dfrac{k_l(\omega)}{Q_l(\omega)} \, \mathrm{Sgn}(\omega)$,

where $\dfrac{1}{Q_l(\omega)} \equiv \sin |\epsilon_l(\omega)|$.

(Use Q_l , not Q . Simply Q is the *seismic* factor. They differ at low ω .)

For any realistic rheology $\bar{J}(\omega)$, the resulting $\dfrac{k_l(\omega)}{Q_l(\omega)} \, \mathrm{Sgn}(\omega)$

has the shape of a kink:



Each $lmpq$ term of the tidal torque has this shape,

because it contains a factor $\dfrac{k_l(\omega)}{Q_l(\omega)} \, \mathrm{Sgn}(\omega) = \mid \bar{k}_l(\omega) \mid \sin \epsilon_l(\omega)$,

where $\omega \; = \; \omega_{lmpq} = \; (l - 2p + q) \, n \; - \; m \; \dot{\theta}$.

Resonances are crossed smoothly:

as $\omega = \omega_{lmpq}$ transcends zero, so does $\mid \bar{k}_l(\omega) \mid \sin \epsilon_l(\omega)$.

The central slope is steep but continuous, for *any* realistic rheology.

The above figure corresponds to the Maxwell body.

A very different rheology in Ferraz Mello arXiv:1204.3957 yields a similar shape.

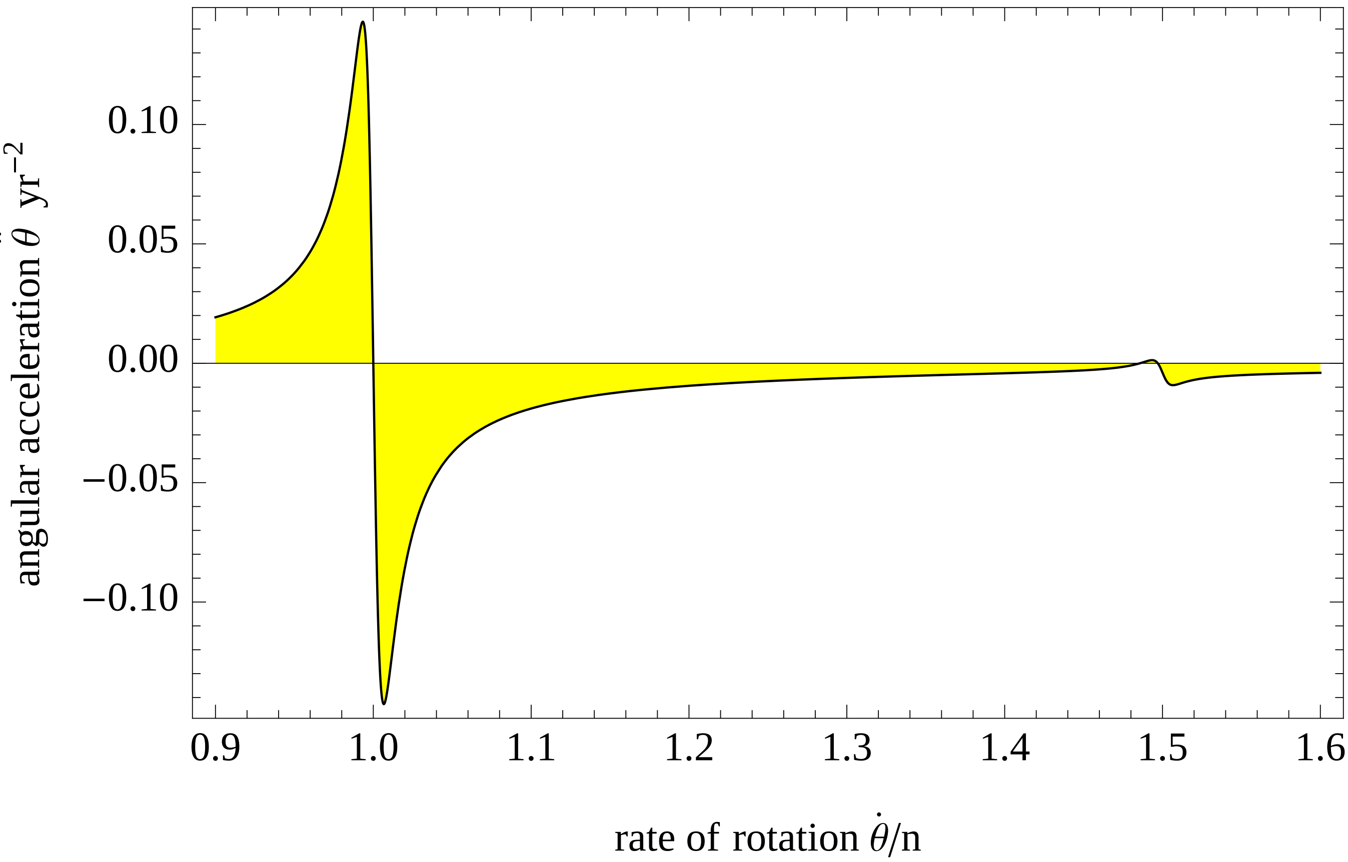
The $lmpq$ term of the tidal torque contains the kink

$$|\,\bar{k}_l(\omega)\,|\,\sin\epsilon_l(\omega) \;=\; |\,\bar{k}_l(\omega_{lmpq})\,|\,\sin\epsilon_l(\omega_{lmpq})\;,$$

where the Fourier modes are $\omega_{lmpq} = (l-2p+q)n - m\,\dot{\theta}$.

Thus each term is a function of the spin rate $\dot{\theta}$.

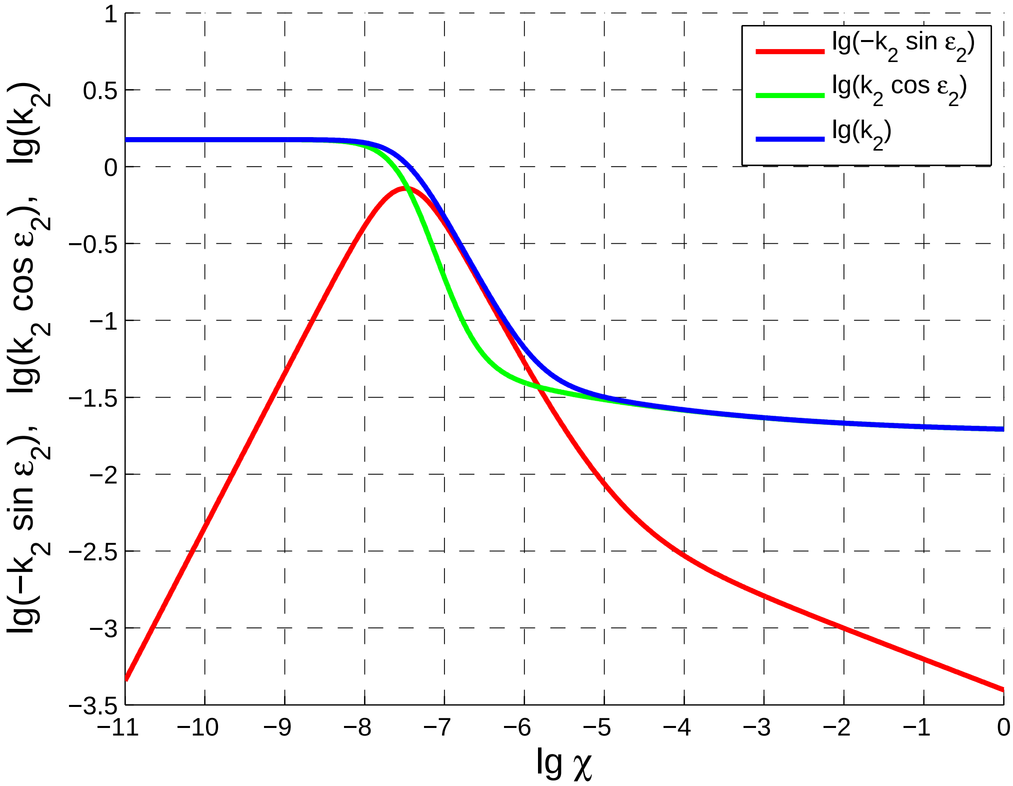
So the overall torque can be treated as a function of $\dot{\theta}$:



The peaks look sharp, but in fact they are continuous, i.e., with no singularity.

The central slope is slightly inclined, not vertical.

The right half of the kink, taken with an opposite sign and in a logarithmic scale (Efroimsky CMDA 2012):



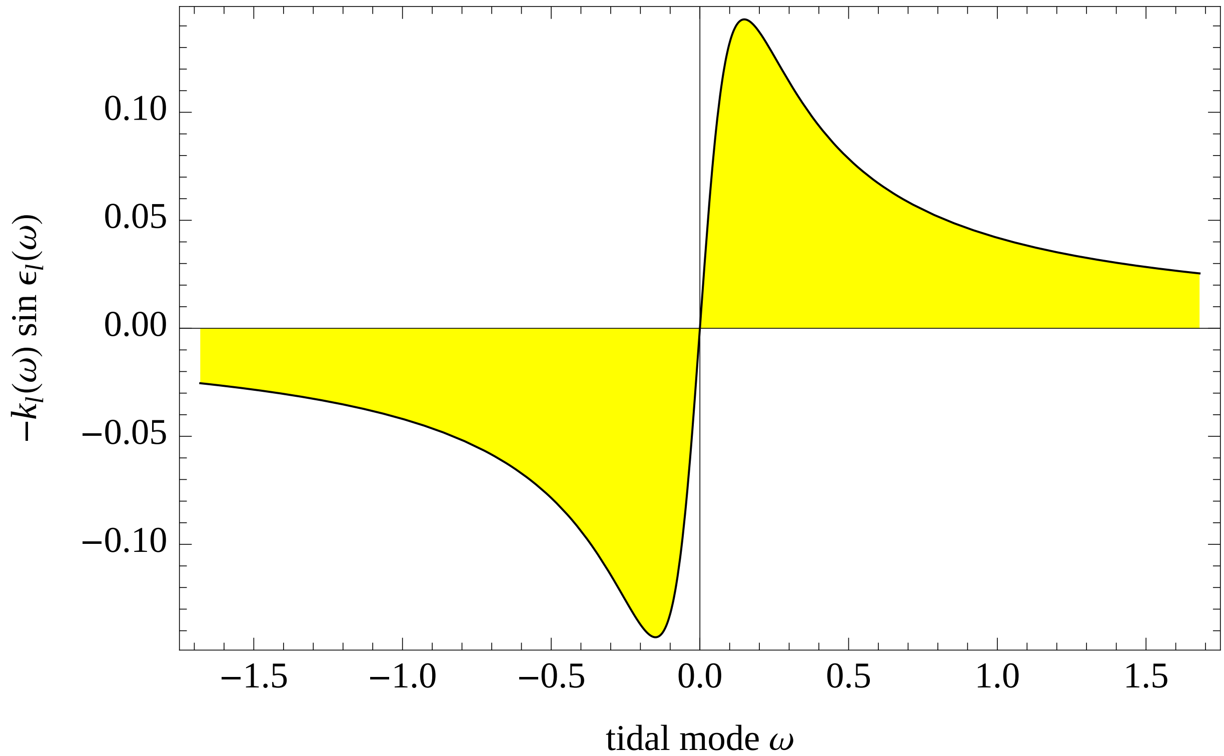
The argument $\chi \equiv |\omega|$ is the physical forcing frequency of the tidal stress in the mantle.

— the frequency-dependence of the Love number $k_2(\omega) \equiv |\bar{k}_2(\omega)|$

— the frequency-dependence of its real part $k_2(\omega) \cos\epsilon_2(\omega) = |\bar{k}_2(\omega)| \cos\epsilon_2(\omega)$

— the frequency-dependence of its imaginary part: $-k_2(\omega) \sin\epsilon_2(\omega) = -|\bar{k}_2(\omega)| \sin\epsilon_2(\omega)$

At what frequencies are the peaks located?



Maxwell time: $\tau_{_M} = \eta/\mu$.   Love numbers:

$$k_{_l} \;=\; \frac{3}{2(l-1)}\;\frac{1}{1\,+\,A_{_l}}\;\;,\qquad A_{_l} \;=\; \frac{3(2l^2+4l+3)\,\mu}{4l\pi G\rho^2 R^2}\quad .$$

For a homogeneous near-spherical Maxwell body:

$$\omega_{_{peak}} \;=\; \pm\;\frac{1}{\tau_{_M}\,A_{_l}} \;=\; \pm\;\frac{4l\pi G\rho^2 R^2}{3(2l^2{+}4l{+}3)\,\eta}\;\;,$$

with no rigidity dependence.

For $l = 2$: $\omega_{_{peak}} \;=\; \pm\;\frac{1}{\tau_{_M}\,A_2} \;=\; \pm\;\frac{8\pi G\rho^2 R^2}{57\,\eta}\;\;,$

G is the Newton gravity constant,

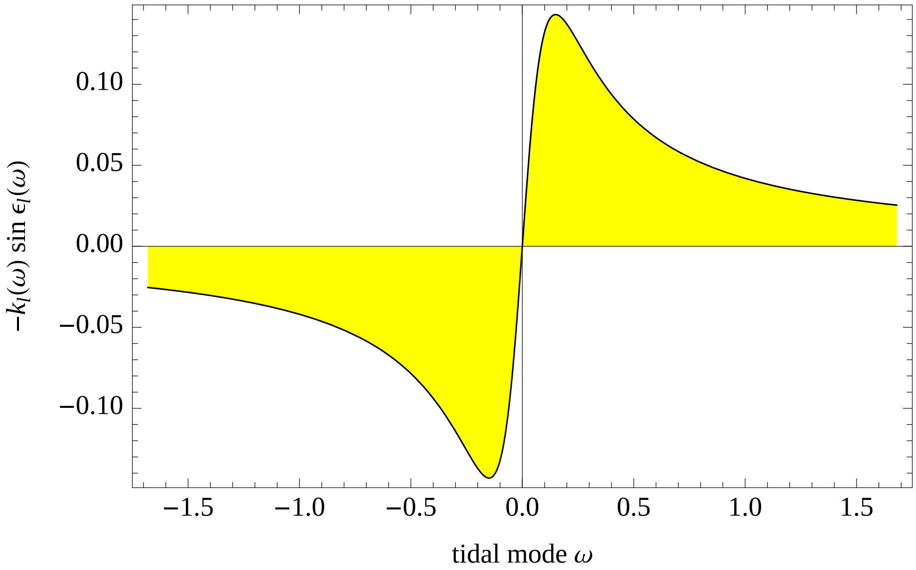
η is the viscosity.
Now, the Moon.

Williams et al (2001, 2008) fit the data to $Q \sim \chi^{p}$. In reality, this was, of course, $Q/k_{_2}$, because in the tidal context $k_{_2}$ and $Q = Q_{_2}$ are inseparable.

Write the scaling law as $\dfrac{k_2}{Q_2} = k_2 \sin|\epsilon_2| \sim \chi^{-p}$.

Williams et al (2008): the slope is almost nil, just a tiny bit on the negative side: $p = -\ 0.09$.

Slightly to the left of the peak



The LLR tidal frequencies were $\approx\ month^{-1}$, so

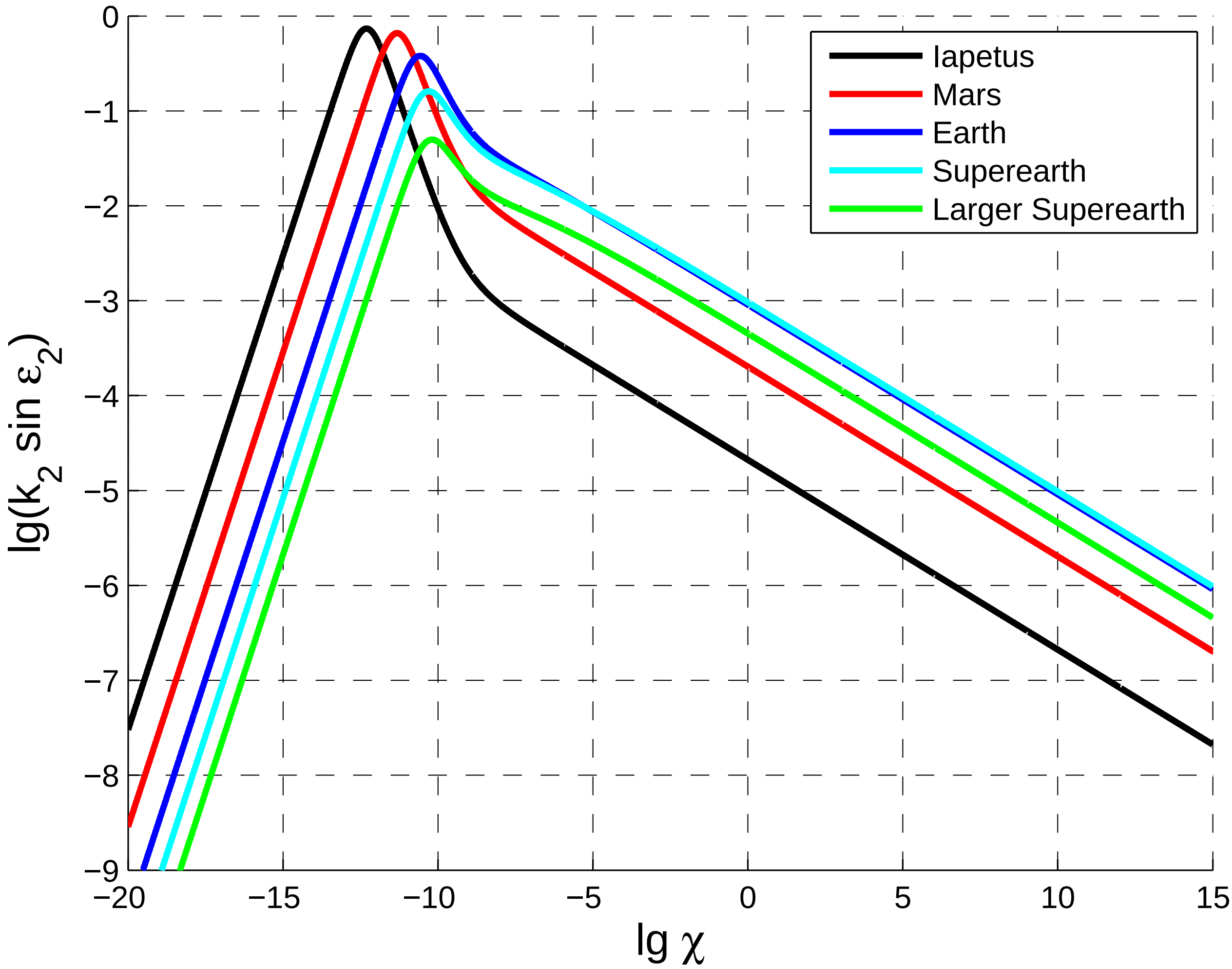
$$\text{month}^{-1} \approx \omega_{peak} = \frac{1}{\tau_M A_2} = \frac{8\pi G\rho^2 R^2}{57\,\eta}$$

Were the Moon homogeneous, it would have

$$\eta = 3 \times 10^{15}\ \text{Pa s}$$

(Partial melt in the low mantle?)

The positive-frequency half of the kink, taken with an opposite sign and in a logarithmic scale, for telluric objects of various sizes (Efroimsky ApJ 2012)



$\chi \equiv |\omega|$ — the positive-definite physical forcing frequency.

If superearths with $R > 4\,R_{Earth}$ exist,

their plots will be well below that of an earth,

because their $k_2(\omega)\,\sin|\epsilon_2(\omega)|$ is much lower due to self-gravitation:

SELF-GRAVITATION ACTS AS EXTRA RIGIDITY, SUPPRESSING THE BULGES.

Spin-orbit interactions of such planets will be suppressed accordingly.

GJ 581d

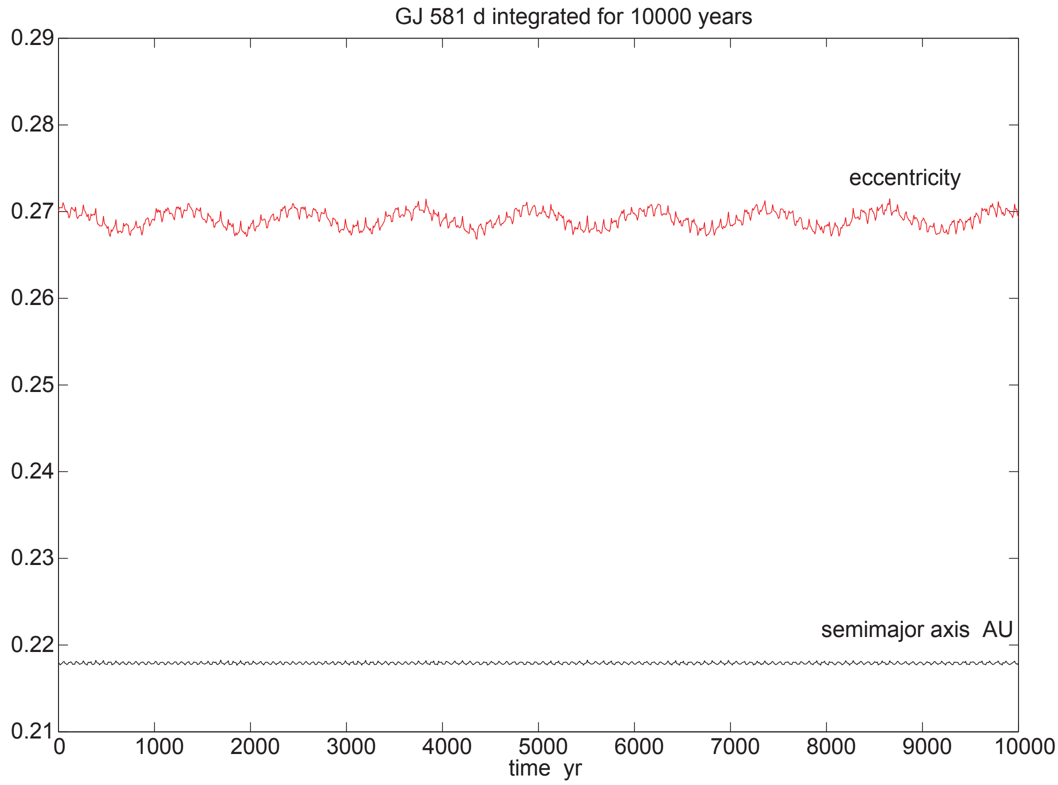
One of the best candidates for habitability

Orbits a nearby M dwarf with a period of 67 days

The system has only 4 planets (Makarov et al. 2012)

Integration shows that the system is dynamically long-term stable but strongly chaotic.

The Lyapunov time for planet *d* is ~30 yr



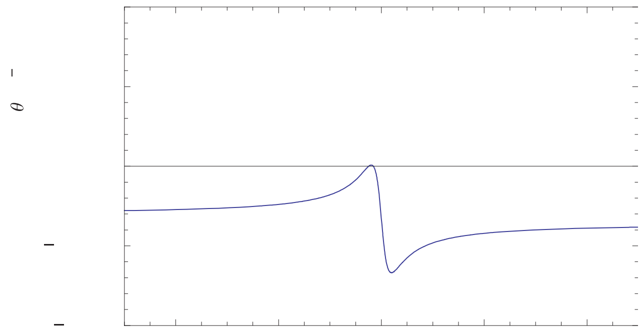
Semimajor axis of GJ 581d is almost constant, eccentricity varying only slightly

Parameters of GJ 581d: data and educated guess

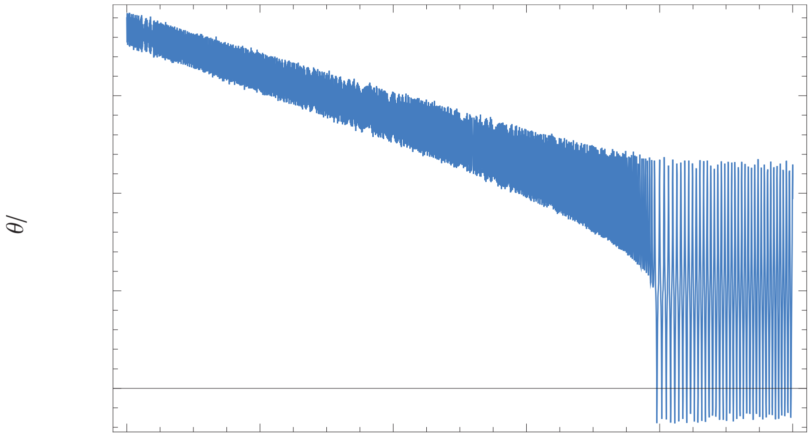
	Parameter	Value
The moment of inertia coefficient	ξ	$\frac{2}{5}$
The radius of the planet	R	$1.7 R_{\mathrm{Earth}}$
	M_{planet}	$7.1 M_{\mathrm{Earth}}$
	M_{star}	$0.31 M_{\odot}$
	a	0.218 AU
	e	0.27
The coefficient of triaxiality	$(B-A)/C$	$5 \cdot 10^{-5}$
	P_{orb}	67 d
The Maxwell time	τ_M	50 yr
The unrelaxed rigidity modulus	μ	$0.8 \cdot 10^{11}$ kg m^{-1} s^{-2}
The Andrade parameter	α	0.2

GJ 581d can be captured in a high-order resonance

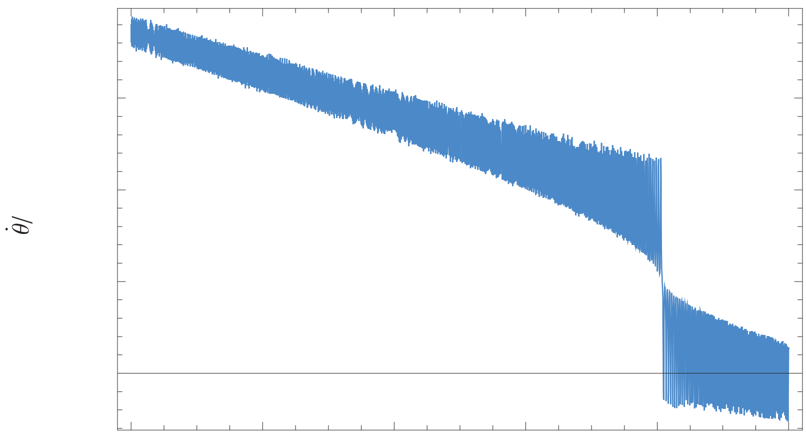
(higher than Mercury)



Tidal torque kink barely crosses 0 at $\dot{\theta}/n = 5/2$



Capture of GJ 581d into 5:2 resonance



GJ 581d traverses 5:2 resonance

From:

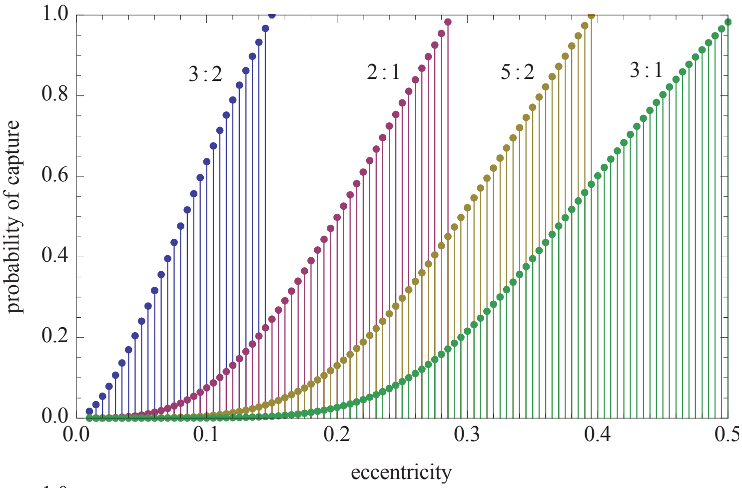
Makarov, Berghea & Efroimsky (2012)

Probabilities of capture

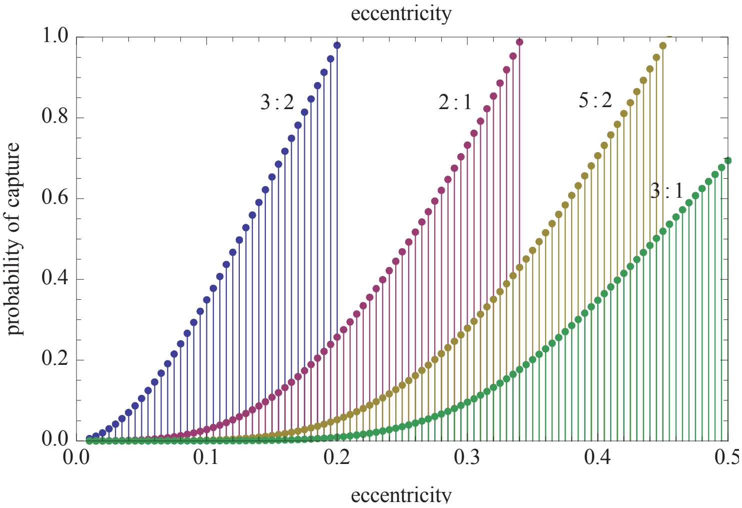
Sensitive to the Maxwell time, i.e., to the temperature.

Less sensitive to triaxiality *(B-A)/C*

For e=0.27, the most probable state of GJ 581d is 2:1 .
This improves the chances for habitability



Probabilities of capture of a warm GJ 581d (Maxwell time = 50 yr)



Probabilities of capture of a cold GJ 581d (Maxwell time = 500 yr)

From:
Makarov, Berghea & Efroimsky (2012)