Polymorphisms and adiabatic chaos

D.Treschev

Steklov Mathematical Institute,

and Moscow State University,
Moscow, Russia

Motivation

Consider the Hamiltonian system (1 + 1/2 dof)

$$\dot{x} = \partial H/\partial y, \quad \dot{y} = -\partial H/\partial x, \quad H = H(x, y, \varepsilon t),$$

H is periodic in $\tau = \varepsilon t$, $\varepsilon \ll 1$.

The action in the frozen system

$$I = I(H, \tau) = \text{area inside the curve } H = \text{const}$$

is an adiabatic invariant. It is smooth outside the separatrix and discontinuous on the separatrix.

$\Delta I \sim \varepsilon \quad$ when $\Delta t \sim 1/\varepsilon$
and the trajectory does not cross the separatrix.

We assume that for any fixed τ the level lines of H (the phase portrait of the "frozen system") are as in the figure.

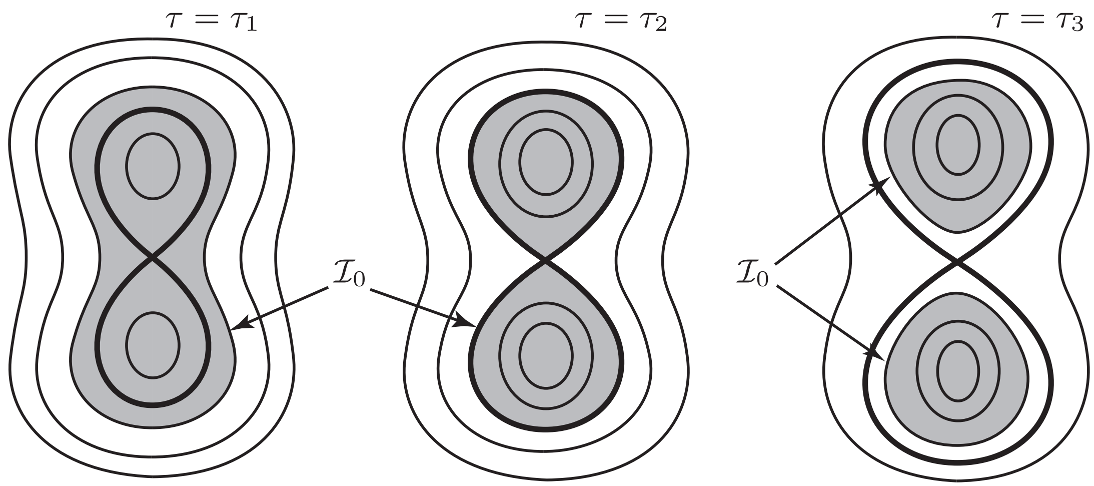


Figure 1: Phase portraits of the "frozen" system

Let $A_+(\tau), A_-(\tau)$, and $A_0(\tau) = A_+(\tau) + A_-(\tau)$ be the areas of the upper, lower loop and the total area of the separatrix.

The initial value $I(\tau_1)$ lies in one of 3 intervals:

$$\Lambda_+ = [0, A_+(\tau_1)], \quad \Lambda_- = [0, A_-(\tau_1)], \quad \Lambda_0 = [A_0(\tau_1), C].$$

We start from $\tau = \tau_1$, $I = \hat{I} := I(\tau_1)$ outside the separatrix. For $\tau = \tau_2$ the area $A(\tau)$ inside the separatrix can become equal to $\hat{I}$. Then for $\tau > \tau_2$ we fall into one of the separatrix loops.

Into which one? For $\varepsilon \to 0$ the answer is: to the upper one with the probability

$$p = p(\tau_2) = \frac{A'_+(\tau_2)}{A'_+(\tau_2) + A'_-(\tau_2)},$$

and to the lower one with the probability $1 - p$. (This is true if $A'_\pm(\tau_2) \geq 0$, otherwise p equals 0 or 1.) At this time moment I jumps, and then it again approximately preserve.

Then the area $A_\pm$ of the loop in which we were captured decreases and at some time moment $t = \varepsilon\tau_3$ the trajectory leaves the loop with $I \approx A_0(\tau_3)$ or $A_\mp(\tau_3)$. The time τ_3 depends on the loop at which we were captured.

When the period passes, I takes some values $\hat{I}_1, \hat{I}_2, \ldots$ with probabilities $p_1, p_2, \ldots$

We obtain the multivalued map $T : \Lambda \to \Lambda$, where Λ is the disjoint union of Λ_+, Λ_-, and Λ_0.

$$T(\hat{I}) = \hat{I}_j \quad \text{with probability } p_j.$$

Remark. If we have a symmetry (areas of the two separatrix loops are the same) then $\hat{I}_j \approx I_1$ for all j.

Because of periodicity in τ we deal with iterations of T. This dynamical system preserves the standard Lebesgue measure on the interval Λ. This means that the system is a *polymorphism*.

Such systems are expected to have generically strong ergodic properties which implies a fast stochastization in the original Hamiltonian system.

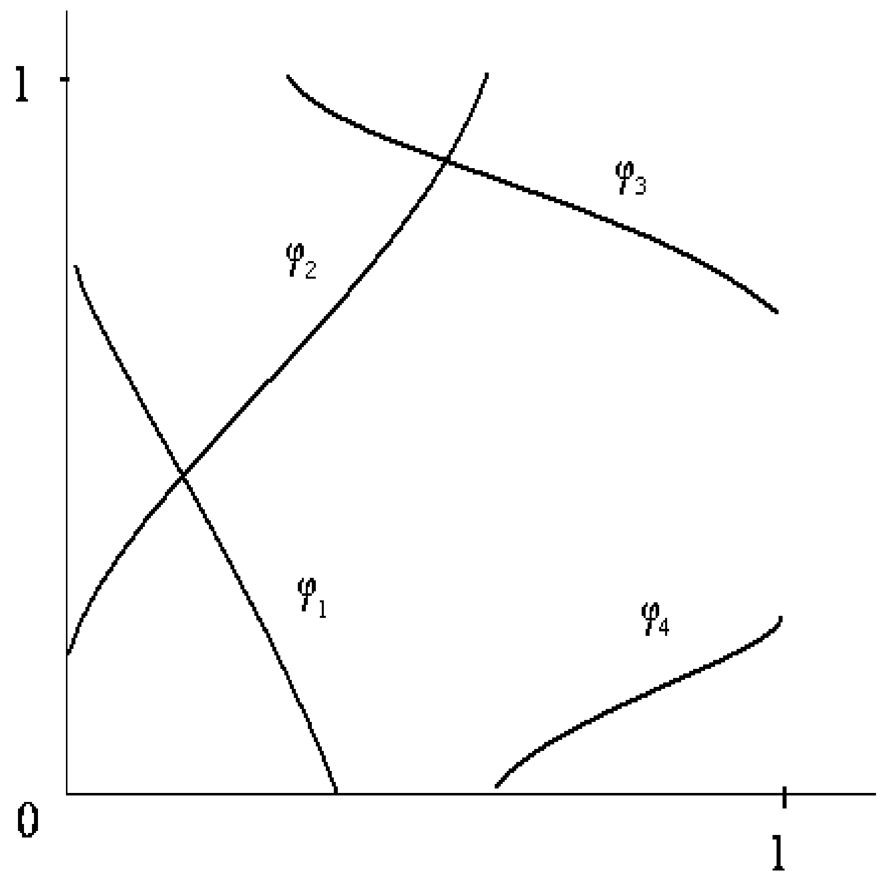
Multivalued self-maps of an interval

Let the interval $[0,1]$ be presented as the union

$$[0,1] = \cup_{j=1}^{J} I_j, \qquad I_j = [a_j, b_j].$$

The intervals I_j can have non-trivial pairwise intersections. For example, it can happen that all I_j equal $[0,1]$. For any j consider the functions

$$\varphi_j : I_j \to [0,1], \quad p_j : I_j \to [0,1].$$



1
φ3
φ2
φ1
φ4
0
1

For any $x \in [0,1]$ we put

$$V(x) = \{j : x \in I_j\}.$$

We assume that the following conditions hold.

P. (Probability) $\sum_{j \in V(x)} p_j(x) = 1$ for any $x \in [0,1]$.

M. (Monotonicity) The functions φ_j are strictly monotone on I_j.

According to **M** there exist the inverse functions $\psi_j = \varphi_j^{-1}$.

Consider the following dynamical system T on $[0,1]$. Any point $x \in I_j$ is mapped to $\varphi_j(x)$ with probability p_j. We denote $T = \big(\varphi_1, \dots, \varphi_J; p_1, \dots, p_J; I_1, \dots, I_J\big)$ or shorter, $T = (\varphi; p; I)$.

The Perron-Frobenius operator

Consider the space $L_2 = L_2([0,1], dx)$, $\langle\, , \rangle$ denotes the corresponding scalar product and $\|\cdot\|$ the L_2-norm.

In a standard way we define the map

$$W_T : L_2 \to L_2, \qquad f \mapsto W_T f.$$

For any $y \in [0,1]$ we put

$$U(y) = \{j : y \in \varphi_j(I_j)\}.$$

Then $\{\psi_j(y) : j \in U(y)\}$ is the set of all preimages of the point y with respect to T. By definition

$$W_T f(y) = \sum_{j \in U(y)} p_j \circ \psi_j(y)\, |\psi_j'(y)|\, f \circ \psi_j(y).$$

For any measurable set $\Omega \subset [0,1]$ we have:

$$\int_\Omega W_T f(y)\, dy = \sum_{j=1}^{J} \int_{\varphi_j^{-1}(\Omega)} p_j(x)\, f(x)\, dx.$$

The positive cone

$$L_2^+ = \{\rho \in L_2 : \rho \geq 0\}$$

can be associated with the space of densities of measures on $[0, 1]$. We have the obvious inclusion

$$W_T(L_2^+) \subset L_2^+.$$

If $\rho \in L_2^+$, and $\rho = W_T\rho$, the measure ν, $d\nu = \rho\, dx$ is said to be invariant w.r.t. T.

Polymorphisms

We assume that the Lebesgue measure is invariant with respect to T.

L. (Lebesgue) $W_T 1 = 1$.

Any map $(\varphi; p; I)$, satisfying **P**, **M**, and **L**, will be said to be a *polymorphism*. A polymorphism can be regarded as a multivalued self-map of an interval preserving the Lebesgue measure.

Vershik construction

According to A.M.Vershik a polymorphism is the ordered diagram

$$\big([0,1]_x, dx\big) \xleftarrow{\pi_x} \big([0,1]_x \times [0,1]_y, \nu\big) \xrightarrow{\pi_y} \big([0,1]_y, dy\big),$$

where π_x and π_y are projections to the x and y component of the product $[0,1]_x \times [0,1]_y$ and ν is a probability measure such that

$$\pi_x \nu = dx \quad \text{and} \quad \pi_y \nu = dy. \tag{1}$$

From the dynamical viewpoint a polymorphism maps randomly any measurable set $\Lambda \subset [0,1]_x$ to the interval $[0,1]_y$ so that the probability of a measurable set $\Omega \subset [0,1]_y$ equals $\nu(\Lambda \times \Omega)$.

The following construction shows a connection between the presented two definitions of a polymorphism. Suppose that $(\varphi; p; I)$ is a polymorphism in the sense of our definition. Let ν be the following measure, supported on the graphs of the functions φ_j. For any $S \subset [0,1] \times [0,1]$ let $\chi_S : [0,1] \times [0,1] \to \mathbb{R}$ be its indicator:

$$\chi_S(x,y) = \begin{cases} 0 & \text{if } (x,y) \notin S, \\ 1 & \text{if } (x,y) \in S. \end{cases}$$

Then by definition

$$\nu(S) = \sum_{j=1}^{J} \int_{I_j} p_j(x)\, \chi_S(x, \varphi_j(x))\, dx.$$

We have an obvious

Proposition 1 *The measure ν satisfies (1).*

An adjoint polymorphism

For any polymorphism $T = (\varphi; p; I)$ we put

$$K_j = \varphi_j(I_j), \quad q_j(x) = p_j \circ \psi_j(x)\,|\psi_j'(x)|.$$

Proposition 2 $(\psi; q; K)$ *is a polymorphism.*

Proof. $(\psi; q; K)$ is obtained from $(\varphi; p; I)$ if in the Vershik diagram we exchange left and right. ■

We say that $(\psi; q; K)$ is adjoint to T: $(\psi; q; K) = T^*$. Obviously $T^{**} = T$.

Proposition 3 $W_{T^*} = W_T^*$.

Corollary 1 *For any polymorphism T we have: $W_T^* 1 = 1$.*

W_T is a Markov (bistochastic) operator i.e.,

(1) $W_T(L_2^+) \subset L_2^+$,

(2) $W_T 1 = W_T^* 1 = 1$.

Mixing and ergodicity

Definition. *The polymorphism T is said to be ergodic if any fixed point of W_T is a constant.*

The polymorphism T is said to be mixing if for any $f \in L_2$

$$W_T^n f \to \overline{f} = \langle 1, f \rangle \quad \textit{in the weak } L_2\textit{-topology as } n \to \infty.$$

If T is mixing, it is ergodic.

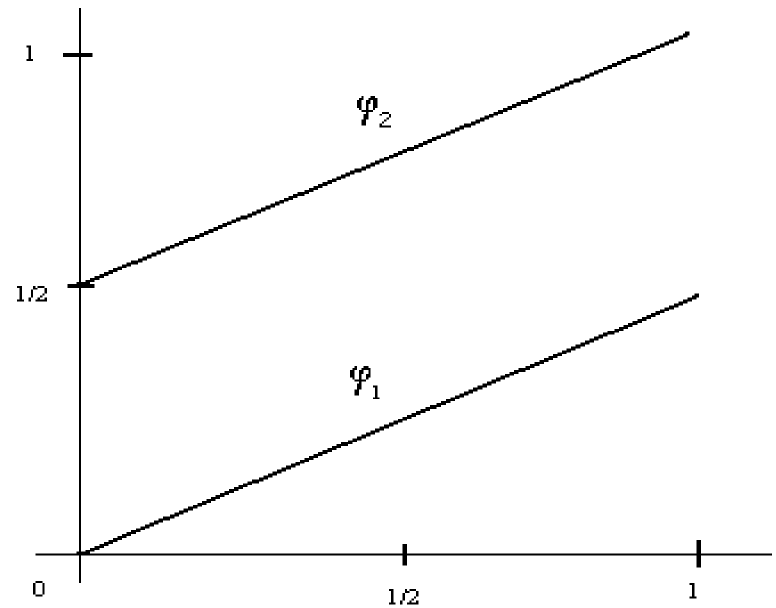
If T is mixing, T^* is also mixing. Indeed,

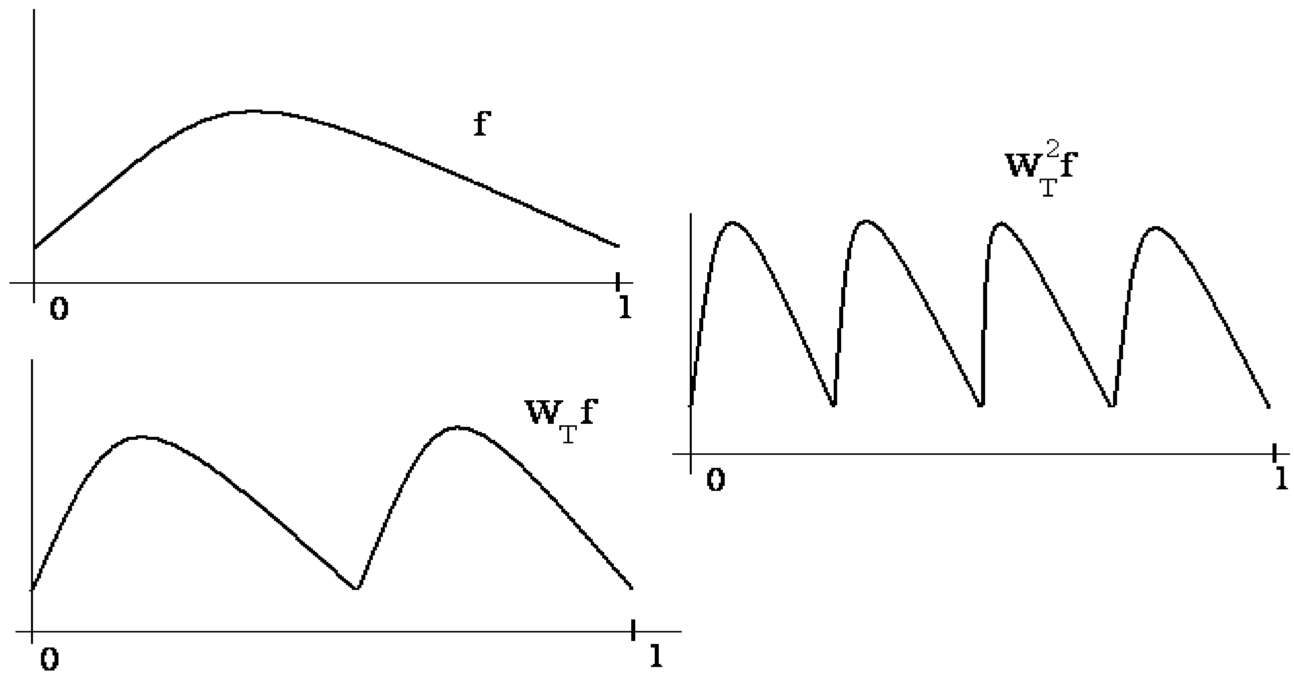
$$\langle W_T^n f, \varphi \rangle \to \overline{f}\,\overline{\varphi} \text{ for any } f, \varphi \iff \langle (W_T^*)^n f, \varphi \rangle \to \overline{f}\,\overline{\varphi} \text{ for any } f, \varphi.$$

An example

The map $T = (\varphi, p)$, where $J = 1,\ \varphi(x) = 2x \bmod 1,\ p = 1$ is a polymorphism because φ preserves the Lebesgue measure.

The adjoint polymorphism $T^* = (\varphi_1, \varphi_2; 1/2, 1/2)$, see the figure:





W_{T^*} acts as shown in the figure $\Rightarrow$ T^* is mixing.

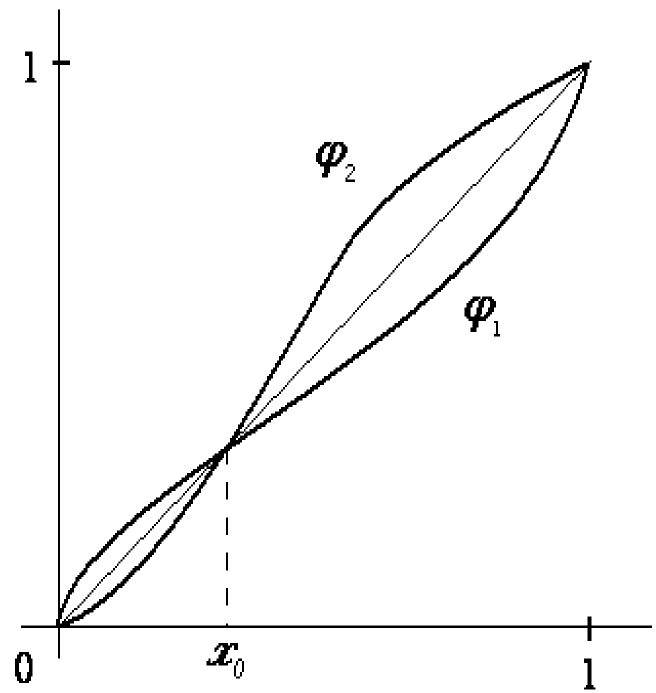
Some non-mixing polymorphisms

Consider polymorphisms T with

$$I_1 = \ldots = I_J = \varphi_1(I_1) = \ldots = \varphi_J(I_J) = [0,1].$$

In this case we use the shorter notation $T = (\varphi; p)$.

Example 1 *Let* $T = (\varphi_1, \varphi_2; p_1, p_2)$ *be a polymorphism such that the functions* φ_1, φ_2 *are increasing and for some* $x_0 \in (0,1)$ $\varphi_1(x_0) = x_0$. *Then* $\varphi_2(x_0) = x_0$ *and the intervals* $[0, x_0]$ *and* $[x_0, 1]$ *are invariant.*


1
φ2
φ1
0
x0
1

Proposition 4 *There exist* $f, p : [0,1] \to [0,1]$ *such that*

a. $f \in C^\infty([0,1])$,

b. $f(x) < x$ *for any* $x \in (0,1)$,

c. $f(0) = 0,\ f(1) = 1$,

d. $f'|_{[0,1]} > 0$,

e. $0 < p(x) < 1$ *for any* $x \in (0,1)$,

f. $(f, f^{-1}; p, 1-p)$ *is a polymorphism.*

The polymorphism $(f, f^{-1}; p, 1-p)$ is not ergodic.

Indeed, let $[\alpha, \beta)$ be a "fundamental" semi-interval i.e.,

$$f^n([\alpha,\beta)) \cap f^m([\alpha,\beta)) = \emptyset \quad \text{for all integer } m \neq n,$$
$$\text{and} \quad \cup_{n\in\mathbb{Z}} f^n([\alpha,\beta)) = (0,1).$$

For any $\rho_0 \in L_2([0,1])$, supp$\rho_0 \subset [\alpha, \beta]$ obviously

$$\rho := \sum_{n=-\infty}^{\infty} \rho_0 \circ f^n \in L_2([0,1]) \quad \text{and} \quad W_T\rho = \rho.$$

Theorem 1 *Let* $T = (\varphi; p)$ *be an ergodic polymorphism such that* $p_j \circ \psi_j(x)\, \psi_j'(x) > c_0 > 0$ $(1 \le j \le J)$ *and the functions* $h_{jk} = \psi_j \circ \varphi_k$ *satisfy conditions (i)–(iii) (see below). Then* T *is mixing.*

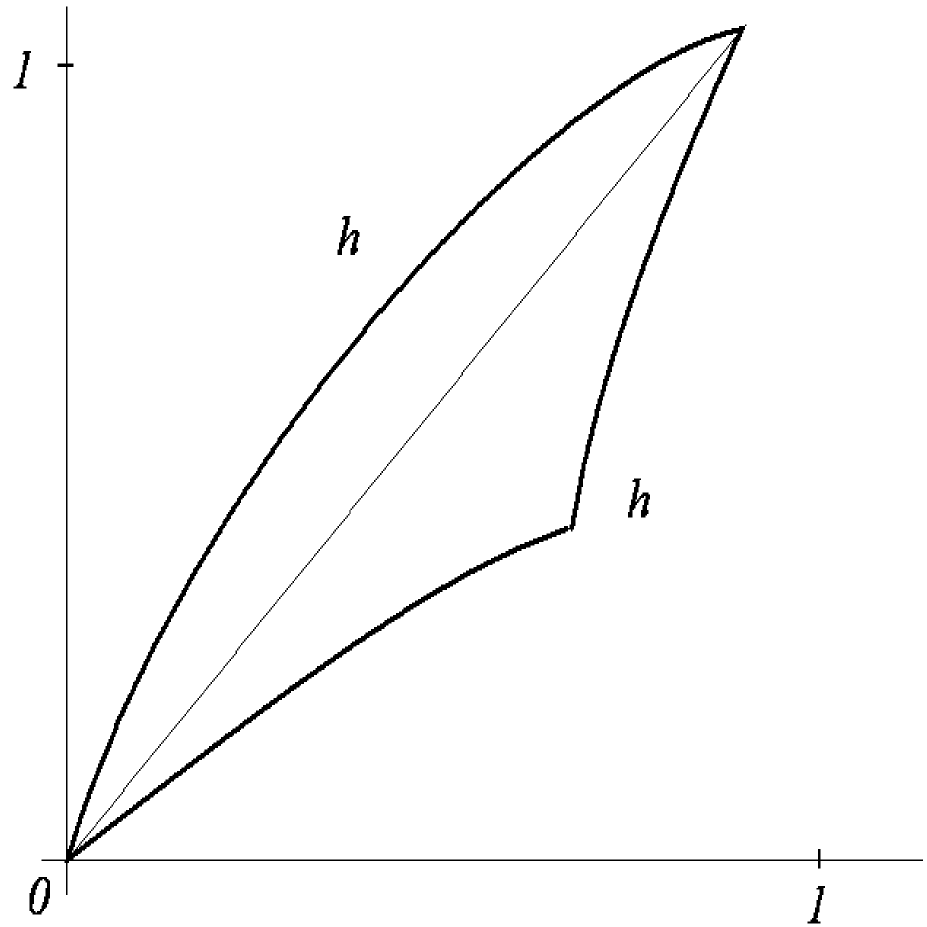
(i) $h : [0, 1] \to [0, 1]$ is smooth of piecewise smooth, $h(0) = 0$, $h(1) = 1$, and both right and left derivatives $h' > 0$ on $[0, 1]$,

(ii) the function $h(x) - x$ does not have zeros on $(0, 1)$,

(iii) $\lim_{x\to 0} h'(x) = \lambda_0$, $\lim_{x\to 1} h'(x) = \lambda_1$, where

$$0 < \lambda_0 < 1 < \lambda_1 \quad \text{or} \quad 0 < \lambda_1 < 1 < \lambda_0.$$

Typical functions h, satisfying (i)–(iii) are presented on the figure:


1
h
h
0
1

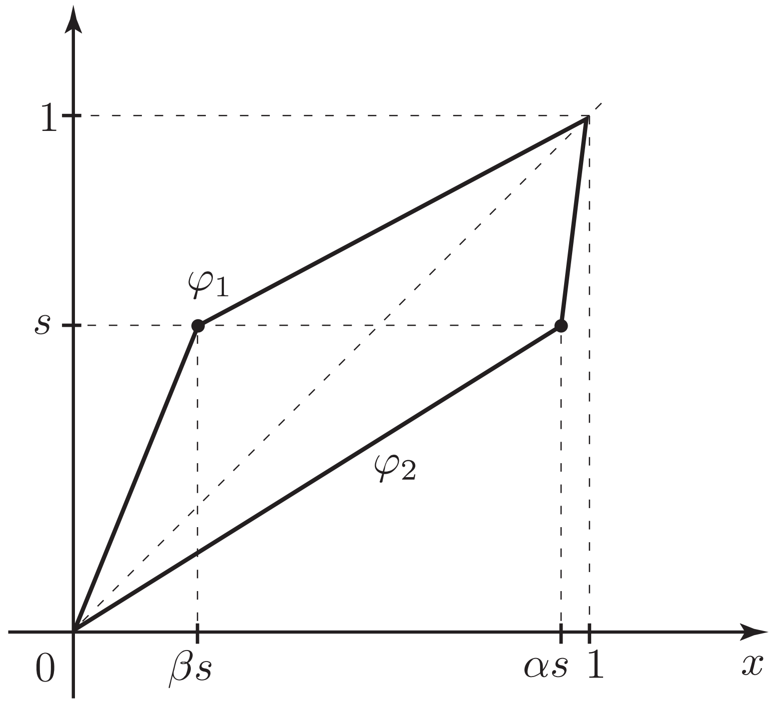


Figure 2: The functions φ_1 and φ_2.

Polymorphisms $T_{\beta,s}$

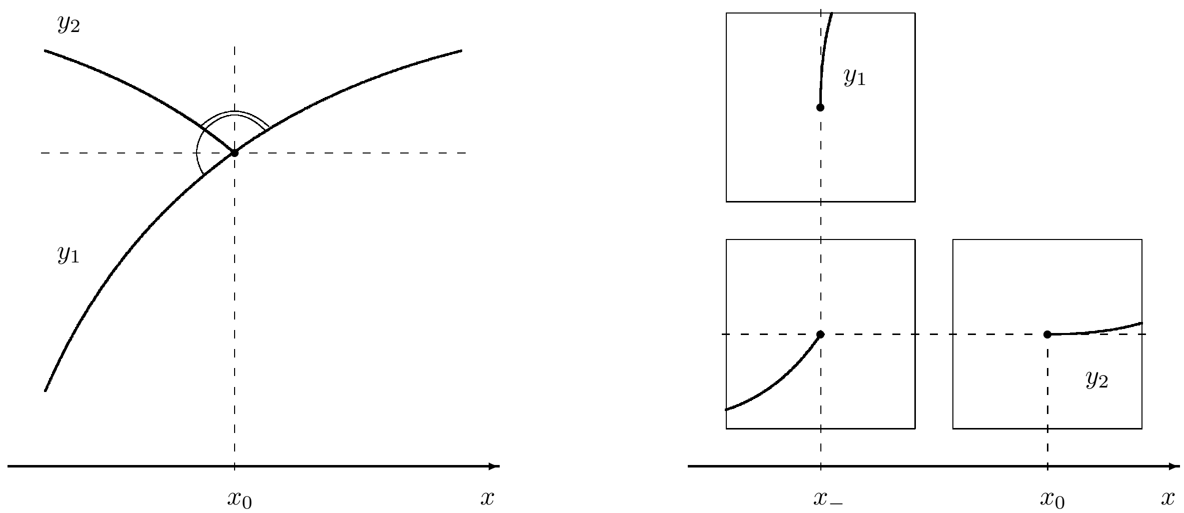
$$T_{\alpha,\beta,s} = (\varphi_1, \varphi_2; p, 1-p), \ \ s \in (0,1), \quad 0 < \beta < \alpha < 1/s.$$

Theorem 2 *$T_{\alpha,\beta,s}$ is ergodic.*

Typical singularities

Theorem 3 *Typical singularities of an "adiabatic" polymorphism are of 3 types:*

(1) Singularities of "joints",
(2) T-crossing,
(3) 3 rays.



a) a)

Figure 3: Singularities of types (2) and (3)

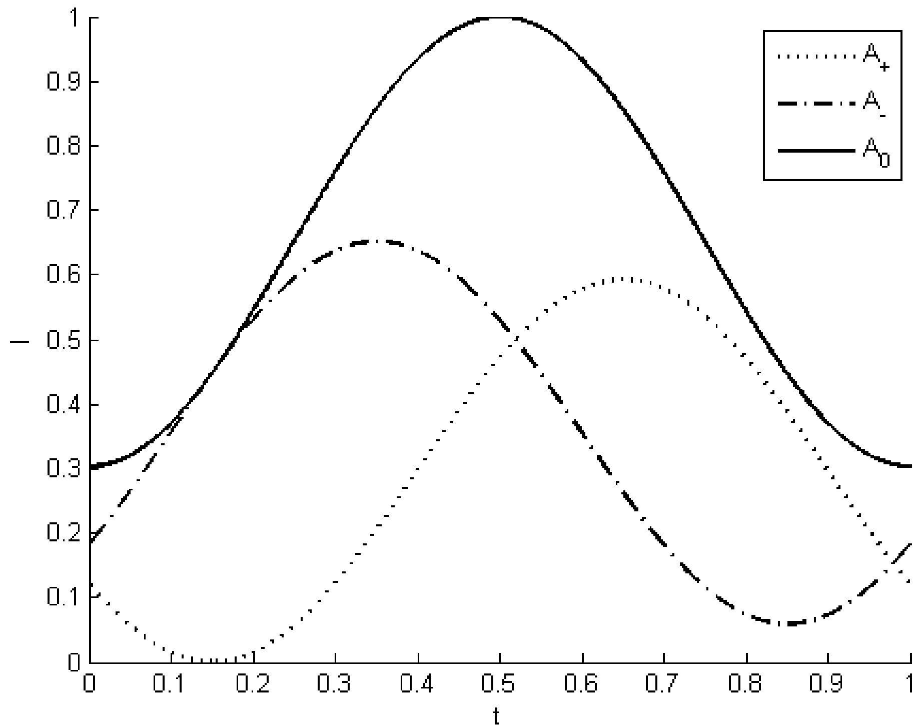


Figure 4: Example: functions A_+, A_-, A_0

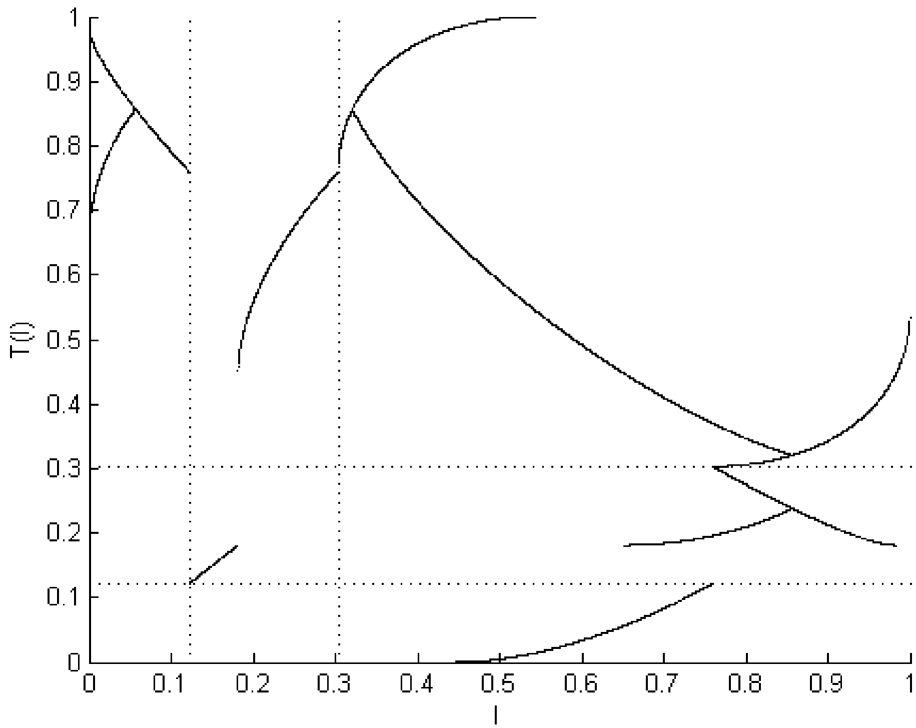


Figure 5: Example: the corresponding polymorphism

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