Families of Periodic Orbits in the Restricted Three Body Problem Near $\mathcal{L}_4$

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Observed Trojan Satellites
http://www.minorplanetcenter.net

- Data from Center for Minor Planets, Nov 28, 2012
- Number of objects at $\mathcal{L}_4$: 3416 (Greeks)
  - increased by 95 since January 2012
  - the Trojan Hector (number 624) is among the Greeks
  - 145 objects at $\mathcal{L}_4$ have names assigned to them
- Number of objects at $\mathcal{L}_5$: 2015 (Trojans)
  - increased by 261 since January 2012
  - The Greek Patroclus (number 617) is among the Trojans
  - 96 objects at $\mathcal{L}_5$ have names assigned to them
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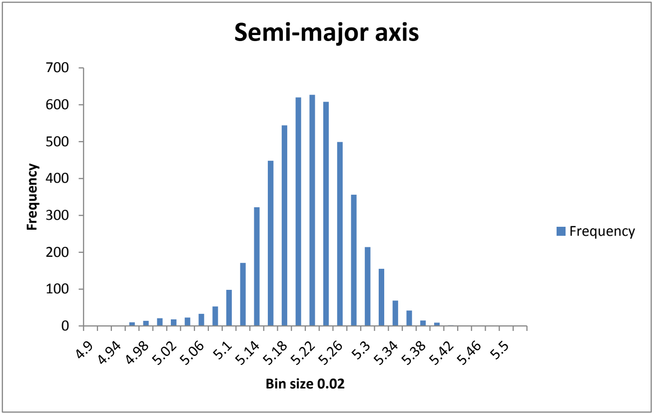
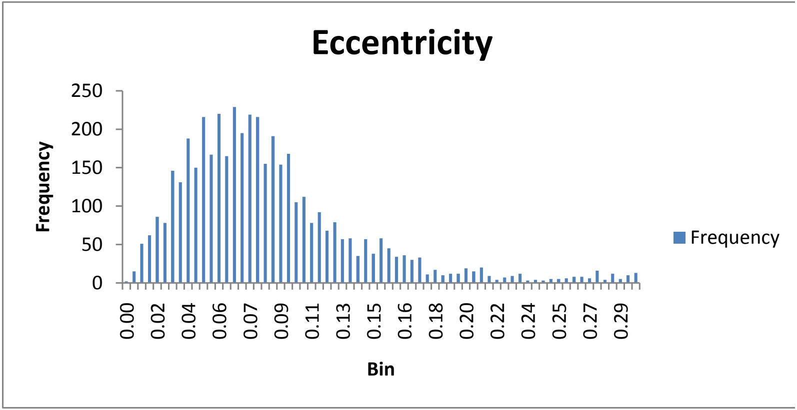
Distribution of orbital elements for Trojan Satellites

Semi major axis: 4.952 to 5.419, Jupiter's value is 5.203,363,01

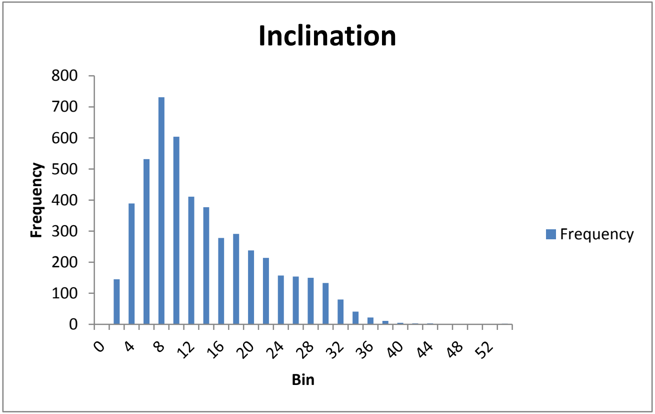


Distribution of the eccentricities

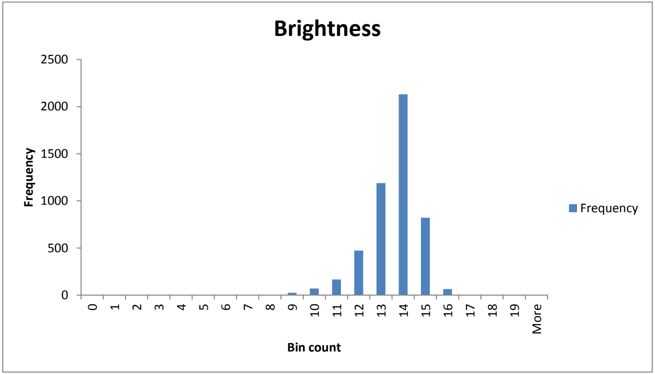
Jupiter's orbit has an eccentricity of 0.048,392,66



The orbit of Jupiter has an inclination of 1.3053 degrees



Distribution of the observed brightness



Simulation of motion in MATLAB

- Data taken from Minor Planet Center
- Orbits are displayed with given orbital elements
- The motion as shown is not valid for longer period of times
- Program does not use numerical integration

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The Circular Restricted Problem of Three Bodies

- In a rotating frame the Hamiltonian of the circular restricted problem of three bodies is

$$H = \frac{1}{2}(y_1^2 + y_2^2) - x_1 y_2 + x_2 y_1 - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2} \ ,$$

- (x_1, x_2) are the components of the position vector in the rotating frame centered at the center of mass
- (y_1, y_2) are their conjugate momenta,
- $r_1 = \sqrt{(x_1+\mu)^2 + x_2^2}$ and $r_2 = \sqrt{(x_1+\mu-1)^2 + x_2^2}$ are the distances of the test particle to the two primaries.

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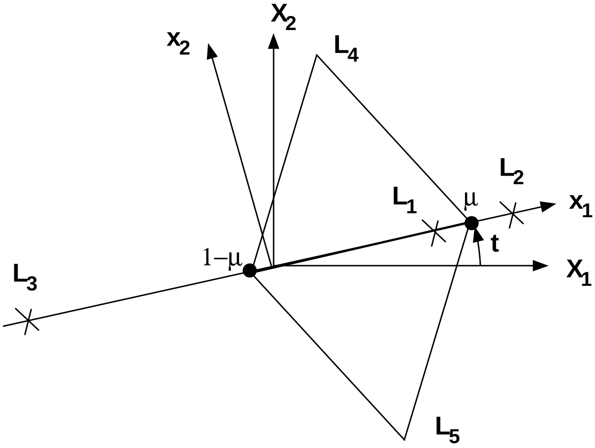
The rotating frame of reference of the restricted problem and the location of the equilibrium points



Linearized System near $\mathcal{L}_4$

- Near equilibrium point $\mathcal{L}_4$

$$\begin{aligned} x_1 &= 1/2 - \mu + X_1 & \qquad y_1 &= -\sqrt{3}/2 + Y_1 \\ x_2 &= \sqrt{3}/2 + X_2 & \qquad y_2 &= 1/2 - \mu + Y_2 \end{aligned}$$

- Expand Hamiltonian function $\quad H = \sum_{k=0} H_k$
- Since $H_1 = 0$ the linearized system is

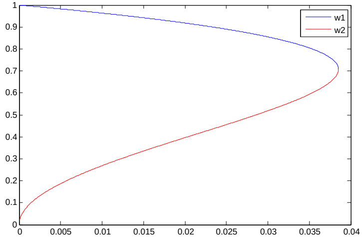
$$\dot{Z} = J\frac{\partial H_2}{\partial Z} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -\frac{1}{4} & \frac{\gamma}{4} & 0 & 1 \\ \frac{\gamma}{4} & \frac{5}{4} & -1 & 0 \end{bmatrix} Z$$

- Abbreviation used: $\gamma = 3\sqrt{3}(1 - 2\mu)$
- For $0 \leq \mu \leq \mu_1$ with $\mu_1 = \frac{1}{2}(1 - \sqrt{\frac{23}{27}})$ the eigenvalues are purely imaginary $\quad \pm i\omega_1, \pm i\omega_2$

The frequencies in $0 \leq \mu \leq \mu_1$

$$\begin{aligned}\omega_1 &= \sqrt{(1+\sqrt{1-27\mu(1-\mu)})/2} \\ &= 1 - \frac{27\mu}{8} - \frac{3213\mu^2}{128} + \cdots\end{aligned}$$

$$\begin{aligned}\omega_2 &= \sqrt{(1-\sqrt{1-27\mu(1-\mu)})/2} \\ &= \frac{3\sqrt{3\mu}}{2}\left(1 + \frac{23\mu}{8} + \frac{4439\mu^2}{128} + \cdots\right)\end{aligned}$$



Problem at $\mu = 0$ (vertical tangent)
Also problem at μ_1 due to repeated eigenvalues when $\omega_1 = \omega_2 = \frac{\sqrt{2}}{2}$

Symplectic linear transformation for $0 < \mu < \mu_1$

$$\begin{bmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{bmatrix} = RS \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}$$

$$R = \begin{bmatrix} \gamma & \gamma & 8\sqrt{\omega_1} & -8\sqrt{\omega_2} \\ -3-4\omega_1^2 & -7+4\omega_1^2 & 0 & 0 \\ 3-4\omega_1^2 & -1+4\omega_1^2 & \gamma\sqrt{\omega_1} & -\gamma\sqrt{\omega_2} \\ \gamma & \gamma & (5-4\omega_1^2)\sqrt{\omega_1} & -(1+4\omega_1^2)\sqrt{\omega_2} \end{bmatrix}$$

$$S = \begin{bmatrix} \frac{1}{2\sqrt{\omega_1(-1+2\omega_1^2)(3+4\omega_1^2)}} & 0 & 0 & 0 \\ 0 & \frac{1}{2\sqrt{\omega_2(7-4\omega_1^2)(-1+2\omega_1^2)}} & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{(-1+2\omega_1^2)(3+4\omega_1^2)}} & 0 \\ 0 & 0 & 0 & \frac{1}{2\sqrt{(7-4\omega_1^2)(-1+2\omega_1^2)}} \end{bmatrix}$$

Linearized system in normal form, but the transformation is singular at $\mu = 0$ and $\mu = \mu_1$

- $$H = \frac{\omega_1}{2}(x_1^2 + y_1^2) - \frac{\omega_2}{2}(x_2^2 + y_2^2)$$
- Two harmonic oscillators with frequencies ω_1 and ω_2
- The Hamiltonian function is indefinite
- $\omega_1 \geq \omega_2$
- **Short period family:** $x_2 = y_2 = 0$ gives $\ddot{x}_1 + \omega_1^2 x_1 = 0$ periodic solution with period $2\pi/\omega_1$
- **Long period family:** $x_1 = y_1 = 0$ gives $\ddot{x}_2 + \omega_2^2 x_2 = 0$ periodic solution with period $2\pi/\omega_2$

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The Variational Equations near $\mathcal{L}_4$

- The differential equations in rotating coordinates

$$\begin{aligned} \ddot{x}_1 - 2\dot{x}_2 &= \Omega_{x_1} \\ \ddot{x}_2 + 2\dot{x}_1 &= \Omega_{x_2} \end{aligned}$$

with $\Omega(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2}$

- The variational equations near $\mathcal{L}_4$: $x_1 = \frac{1}{2} - \mu + \xi_1,\ x_2 = \frac{\sqrt{3}}{2} + \xi_2$

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- Terms on right hand side can be put into diagonal form

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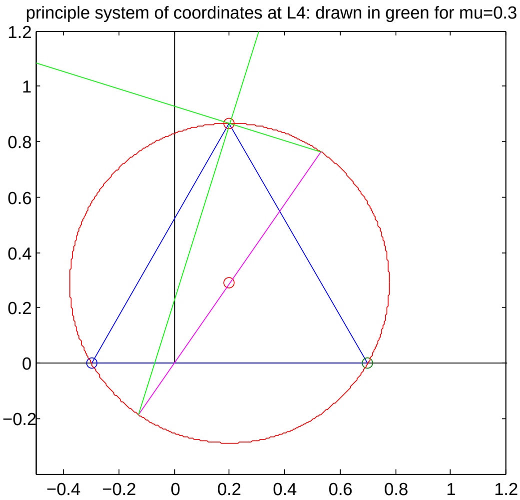
Construction of the Principle Frame near $\mathcal{L}_4$ rotation by angle α with $\tan 2\alpha = -\sqrt{3}(1-2\mu)$



Solution to transformed variational Equations

- $\xi = T\bar{\xi}$ is a rotation in the ξ_1–ξ_2 plane. It results in equations of the same form

$$\begin{aligned} \ddot{\bar{\xi}}_1 - 2\dot{\bar{\xi}}_2 &= \lambda_1 \bar{\xi}_1 \\ \ddot{\bar{\xi}}_2 + 2\dot{\bar{\xi}}_1 &= \lambda_2 \bar{\xi}_2 \end{aligned}$$

- Form of the solution

$$\begin{aligned} \bar{\xi}_1 &= A_1 \cos \omega_1 t + A_2 \cos \omega_2 t \\ \bar{\xi}_2 &= C_1 \sin \omega_1 t + C_2 \sin \omega_2 t \end{aligned}$$

- Short Period Orbit: Set $A_2 = C_2 = 0$ and find $A_1 = -\frac{2\omega_1}{\omega_1^2 + \lambda_1} C_1$
- Long Period Orbit: Set $A_1 = C_1 = 0$ and find $A_2 = -\frac{2\omega_2}{\omega_2^2 + \lambda_1} C_2$
- Note $|A_1/C_1| > 1$ and also $|A_2/C_2| > 1$ so that semi major axes of both ellipses are along $\bar{\xi}_1$ axis
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- $\xi = T\bar{\xi}$ is a rotation in the ξ_1–ξ_2 plane. It results in equations of the same form

$$\begin{aligned} \ddot{\bar{\xi}}_1 - 2\dot{\bar{\xi}}_2 &= \lambda_1 \bar{\xi}_1 \\ \ddot{\bar{\xi}}_2 + 2\dot{\bar{\xi}}_1 &= \lambda_2 \bar{\xi}_2 \end{aligned}$$

- Form of the solution

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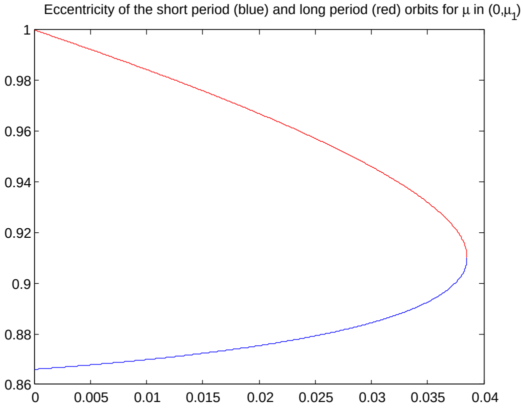
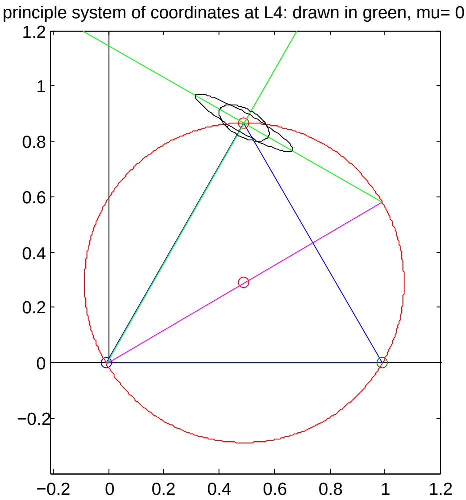
Eccentricity of short and long periodic orbits around $\mathcal{L}_4$ in the orbital plane

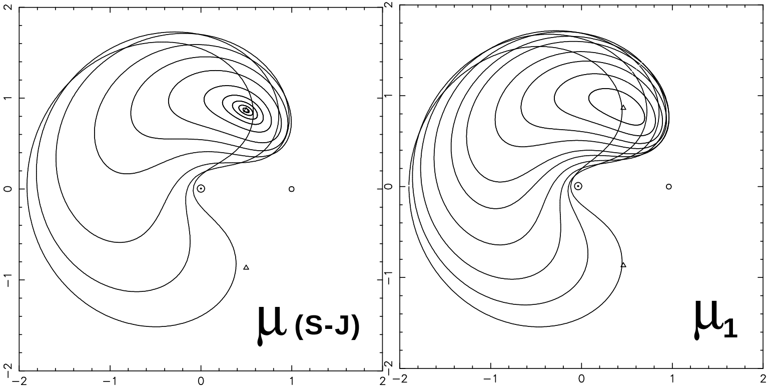
$$e_1 = \sqrt{1 - \frac{(\omega_1^2 + \lambda_2)^2}{4\omega_1^2}} \qquad e_2 = \sqrt{1 - \frac{(\omega_2^2 + \lambda_2)^2}{4\omega_2^2}}$$



Principle Frame with short and long period orbits



Short period orbits



Comments to previous picture

- The family of short period orbits emanating from $\mathcal{L}_4$ is the backbone of the *Trojan web*.
- There is of course a symmetric family emanating from $\mathcal{L}_5$.
- In the inertial frame of reference both families look like a set a moderately perturbed Keplerian orbits with an eccentricity going from zero at $\mathcal{L}_4$ (or $\mathcal{L}_5$) to a rather large value when they meet the symmetric family emanating from $\mathcal{L}_3$.
- The argument of pericenter goes from 60° ahead (or behind for $\mathcal{L}_5$) of the perturbing body to 180° ahead (or behind for $\mathcal{L}_5$).
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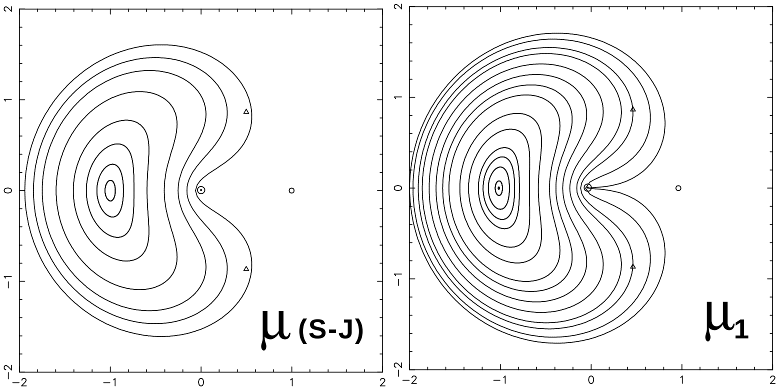
Family of symmetric periodic orbits emanating from $\mathcal{L}_3$



Existence of short periodic orbits

- Since $\omega_1 > \omega_2$ for $0 < \mu < \mu_1$ the short periodic orbits constructed in the linear system will persist in the nonlinear system at least near $\mathcal{L}_4$
- Lyapunov's theorem gives no information about the global behavior of the short period family
- The general theory about the continuation of periodic orbits gives some information
- A natural family of periodic orbits of a two-degree of freedom Hamiltonian system with no other first integral than the Hamiltonian can be continued until one of the following things happens:
  - the orbit tends to infinity;
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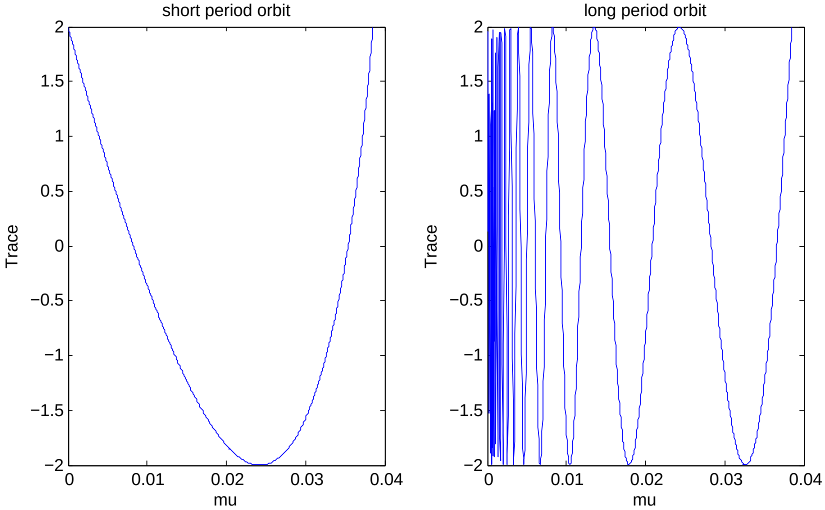
Trace for characteristic exponents



Resonance Cases for $\mu = \mu_{p/q}$ we have $\omega_1/\omega_2 = p/q$

Special values for the mass ratio $\mu = \mu_p$ when $\omega_1/\omega_2 = p$

μ_1	= 0.038520896504551	μ_7	= 0.002912184522396
μ_2	= 0.024293897142052	μ_8	= 0.002249196513710
μ^{**}	$\approx$ 0.02072	μ_9	= 0.001787848394744
μ_3	= 0.013516016022453	μ_{10}	= 0.001454405739621
μ^*	= 0.012723988746542	μ_{11}	= 0.001205829591109
μ_{EM}	= 0.01215002	μ_{12}	= 0.001015696721082
μ_d	= 0.010913667677201	μ_{SJ}	= 0.000953875
μ_4	= 0.008270372663897	μ_{13}	= 0.000867085298404
μ_5	= 0.005509202949840	μ_{14}	= 0.000748764338855
μ_6	= 0.003911084259658	μ_{15}	= 0.000653048708761

ω_1/ω_2 rational

- The case ω_1/ω_2 rational is of special interest, since it is the organizing center for many interesting bifurcations and the reason for instabilities.
- For ω_1/ω_2 irrational the normal form would be

$$H(I_1, I_2, \phi_1, \phi_2) = \omega_1 I_1 - \omega_2 I_2 + K(I_1, I_2),$$

  but transformation does not converge
- The solution would be invariant tori: $I_1 = c_1,\ I_2 = c_2$ with linear flows of constant slope
- The question is: Which tori exist, when ω_1/ω_2 is rational and normalization is carried out only to a finite order?
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Normal Form in Action Angle Variables for $\mu = \mu_{p/q}$

- $\mu_{p/q}$ such that $\omega_1/\omega_2 = p/q$ with $p > q$ and $p + q \geq 4$
- $$\begin{aligned} H &= \omega_1 I_1 - \omega_2 I_2 + \varepsilon^2(\lambda_1 I_1 - \lambda_2 I_2 + \frac{A}{2} I_1^2 + B I_1 I_2 + \frac{C}{2} I_2^2) \\ &\quad + \cdots + \varepsilon^{p+q-2} I_1^{p/2} I_2^{q/2} G \cos\psi + \cdots \end{aligned}$$
- $\psi = q\phi_1 + p\phi_2 + \alpha$
- $$\begin{aligned} \dot{I}_1 &= -\varepsilon^{p+q-2} I_1^{p/2} I_2^{q/2} qG \sin\psi + \cdots \\ \dot{I}_2 &= -\varepsilon^{p+q-2} I_1^{p/2} I_2^{q/2} pG \sin\psi + \cdots \\ \dot{\phi}_1 &= -p\lambda - \varepsilon^2(\lambda_1 + AI_1 + BI_2) + \cdots \\ \dot{\phi}_2 &= q\lambda - \varepsilon^2(-\lambda_2 + BI_1 + CI_2) + \cdots \end{aligned}$$

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Periodic Orbits Derived from Normal Form

- Condition for periodic orbit to exist for $\varepsilon \neq 0$:
  - Nontrivial characteristic multipliers can't be 1
  - Otherwise solve the bifurcation equations to get periodic orbit
  - One equation can be replaced by Hamiltonian
  - Use Fredholm alternative theorem to solve equations
  - If an action variable is close to 0 switch to Cartesian coordinates
- Results:
  - **Short period:** $T_1 \approx \frac{2\pi}{p\lambda}$. Family near $I_1 > 0$ and $I_2 \approx 0$
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Periodic Orbits Derived from Normal Form

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The normal form for the restricted three-body problem for $0 < \mu < \mu_1$, and $\mu \neq \mu_2,\ \mu_3$ through fourth order terms

- $$H = \omega_1 I_1 - \omega_2 I_2 + \frac{1}{2}(AI_1^2 + 2\mathrm{BI}_1 I_2 + CI_2^2) + \cdots .$$

- $$A = \frac{\omega_2^2(81 - 696\omega_1^2 + 124\omega_1^4)}{72(1 - 2\omega_1^2)^2(1 - 5\omega_1^2)},$$

  $$B = -\frac{\omega_1\omega_2(43 + 64\omega_1^2\omega_2^2)}{6(1 - 2\omega_1^2)(1 - 2\omega_2^2)(1 - 5\omega_1^2)(1 - 5\omega_2^2)},$$

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Short Period Orbits, $I_1 > 0$ and $I_2 \approx 0$

- Nontrivial characteristic multipliers: eigenvalues of

$$\frac{\partial(x_2(2\pi), y_2(2\pi))}{\partial(x_{20}, y_{20})} = \begin{pmatrix} \cos(2\pi\nu) & \sin(2\pi\nu) \\ -\sin(2\pi\nu) & cos(2\pi\nu) \end{pmatrix} + \mathbf{O}(\varepsilon^3)$$

- and they are $cos(2\pi\nu) \pm i\sin(2\pi\nu)$ with

$$\nu = \frac{q}{p} - \frac{\varepsilon^2}{p^2\lambda}(MJ_1 + \sigma) + \mathbf{O}(\varepsilon^3)$$

- The eigenvalues are on the unit circle and have to stay there for $J_1 \geq 0$ since they are not +1 or -1.
- Note: If $J_1 = -\sigma/M > 0$ and orbit is traveled q times, the characteristic multipliers become +1. This will allow for the bifurcation of another family of periodic orbits.

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Long Period Family $I_2 > 0$ and $I_1 \approx 0$

- Use Φ_2 as new independent variable, and $x_1 = x_1(\Phi_2)$, $y_1 = y_1(\Phi_2)$
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The exceptional resonance case when $q = 1$

- The Jacobian is zero at $\varepsilon = 0$ when $q = 1$
- Expand bifurcation equations in ε and divide by ε^2
- If the modified Jacobian

$$\frac{\partial(\Gamma_2, \Gamma_3)}{\partial(x_{10}, y_{10})} = (NJ_2 + \sigma)^2 \neq 0$$

  then the bifurcation equations can be solved also when $q = 1$
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Bifurcation equations for orbits with period $T = 2\pi/\lambda + \varepsilon\beta$ and $I_1 \neq 0$ and $I_2 \neq 0$

- Use differential equations in action–angle variables
- The initial conditions are $I_1(0) = J_1$, $I_2(0) = J_2$ and $\phi_2(0) = \psi_2$
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$$\Gamma_2 = J_1^{q/2} J_2^{p/2} G \sin(p\psi_2 + \alpha) + \mathbf{O}(\varepsilon^2) = 0 \quad (1)$$
$$\Gamma_3 = MJ_1 + NJ_2 + \sigma + \mathbf{O}(\varepsilon) = 0 \quad (2)$$

- If (2) allows for solutions with $J_1 > 0$ and $J_2 > 0$ we have a torus of periodic solutions
- Periodic orbits are possible on this torus when 1 is satisfied $\sin(p\psi_2 + \alpha) + \cdots = 0$
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- The initial conditions are $I_1(0) = J_1$, $I_2(0) = J_2$ and $\phi_2(0) = \psi_2$
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$$\Gamma_2 = J_1^{q/2} J_2^{p/2} G \sin(p\psi_2 + \alpha) + \mathbf{O}(\varepsilon^2) = 0 \quad (1)$$
$$\Gamma_3 = MJ_1 + NJ_2 + \sigma + \mathbf{O}(\varepsilon) = 0 \quad (2)$$

- If (2) allows for solutions with $J_1 > 0$ and $J_2 > 0$ we have a torus of periodic solutions
- Periodic orbits are possible on this torus when 1 is satisfied $\sin(p\psi_2 + \alpha) + \cdots = 0$
- Two distinct solutions are possible $\psi_2 = -\alpha/p + \cdots$ and $\psi_2 = (\pi - \alpha)/p + \cdots$, one is stable the other unstable
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Equation $MJ_1 + NJ_2 + \sigma = 0$ for $\mathcal{L}_4$

- Detuning:

$$\sigma = \omega_2\lambda_1 - \omega_1\lambda_2 = \omega_2\frac{d\omega_1}{d\mu} - \omega_1\frac{d\omega_2}{d\mu}$$

- For restricted three body problem

$$\sigma = \frac{-3\sqrt{3}(1-2\mu)}{4\sqrt{\mu(1-\mu)(1-27\mu(1-\mu))}} < 0$$

- Since $\sigma < 0$ it corresponds to $\mu > \mu_{p/q}$
- To see what happens for $\mu < \mu_{p/q}$ change sign of σ

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Values for $MJ_1 + NJ_2 + \sigma = 0$ near $\mathcal{L}_4$

- $$M = \frac{\omega_2(324 - 4029\omega_1^2 + 6397\omega_1^4 - 3828\omega_1^6 + 620\omega_1^8)}{72(1-2\omega_1^2)^2(4 - 25\omega_1^2 + 25\omega_1^4)}$$
- $$N = \frac{\omega_1(-516 + 239\omega_1^2 - 1367\omega_1^4 + 1348\omega_1^6 + 620\omega_1^8)}{72(1-2\omega_1^2)^2(4 - 25\omega_1^2 + 25\omega_1^4)}$$
- $N = 0$ for $\mu = \mu^* = 0.01272398874654163$

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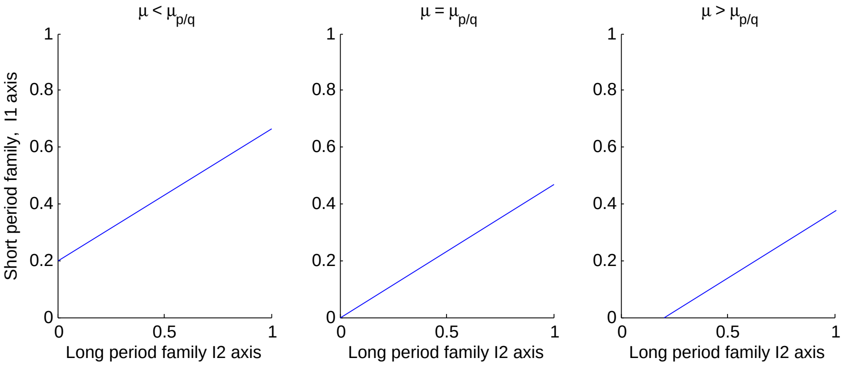
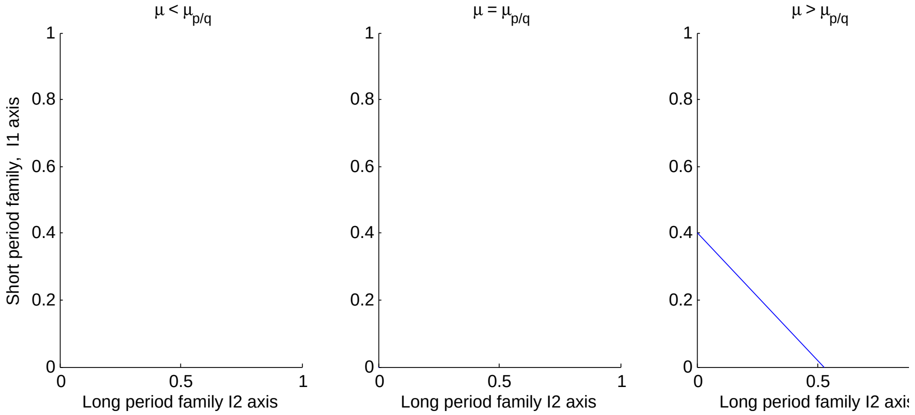
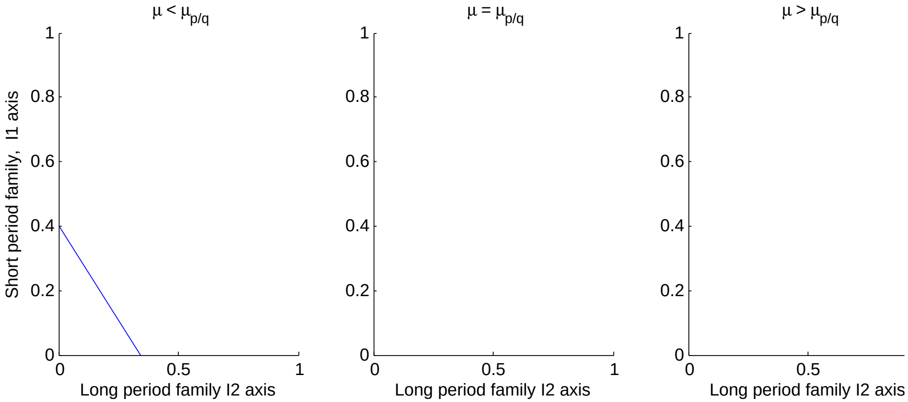
"Open case" for $0 < \mu < \mu^*$



“Bridge” for $\mu^* < \mu < \mu_2 \qquad \mu \neq \mu_3$



"Bridge" for $\mu_2 < \mu < \mu_1$



What happens when $J_1 \to 0$ or $J_2 \to 0$ (case $\mu < \mu^*$)

Theorem (Case $p > q > 2$)

- *For $\mu < \mu_{p/q}$ bifurcation of a stable and unstable family from the short period family (repeated p times)*
- *For $\mu = \mu_{p/q}$ four families of periodic orbits emanate from $\mathcal{L}_4$: Short, long and two with the common period*
- *For $\mu > \mu_{p/q}$ bifurcation of a stable and unstable family from the long period family (repeated q times)*

Result follows from normal form through fourth order terms

What happens when $J_1 \to 0$ or $J_2 \to 0$ (continued)

Theorem (Case $p > q = 2$)

- *For $\mu < \mu_{p/2}$ bifurcation of a stable and unstable family from the short period family (repeated p times)*
- *For $\mu = \mu_{p/2}$ four families of periodic orbits emanate from $\mathcal{L}_4$: Short, long and two with the common period*
- *For $\mu > \mu_{p/2}$ long period family has interval of instability and the two families connect to the end of the interval with orbit traveled twice*

To show result need to have resonance terms, that is $G \neq 0$

What happens when $J_1 \to 0$ or $J_2 \to 0$ (continued)

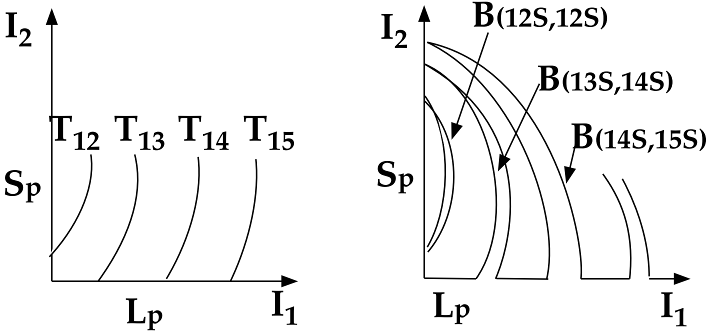
Theorem (Case $p > 3$ and $q = 1$)

- *For $\mu < \mu_p$ bifurcation of a stable and unstable family from the short period family (repeated p times)*
- *For $\mu = \mu_p$ four families of periodic orbits emanate from $\mathcal{L}_4$: Short, and three long period families*
- *For $\mu > \mu_p$ long period family breaks up and connects with the families of the common period*

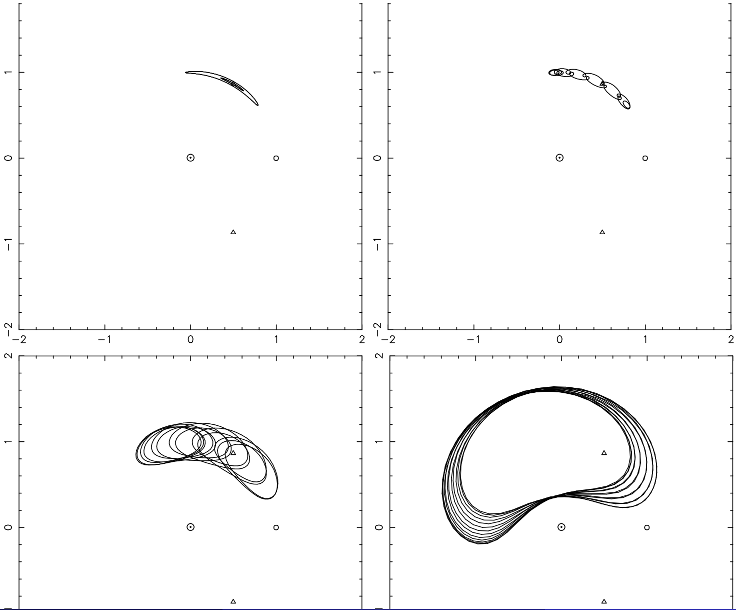
To prove result need to have resonance terms with $G \neq 0$

Schematic presentation of results for μ in interval $[\mu_{13}, \mu_{12}]$

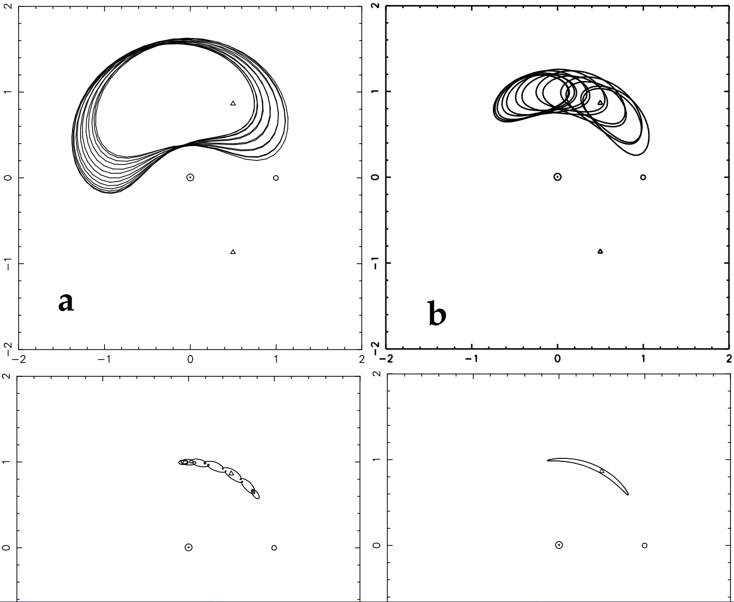
Left panel from terms through order 4
Right panel from normal form through order 14



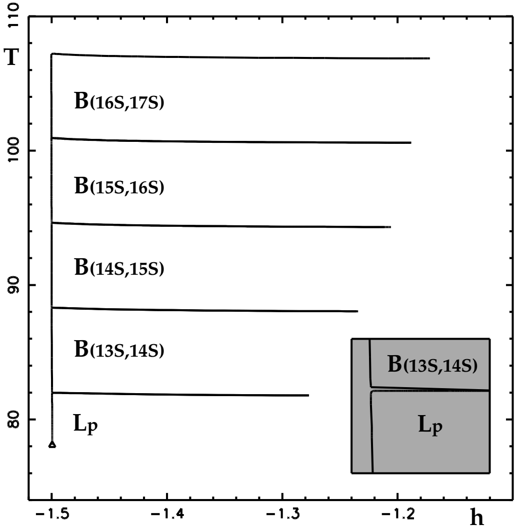
Long period orbits for μ_{SJ}: $\mathcal{B}(L, 13S)$



Short period bridge $\mathcal{B}(13S, 14S)$



Period versus Energy for long period chain $\mathcal{L}p, \mathcal{B}(13S, 14S), \mathcal{B}(1S, 15S), \mathcal{B}(15S, 16S), \mathcal{B}(16S, 17S)$



Fast and slow variables

- Gelfreich & Lerman (2002): *'Long periodic orbits and invariant tori in singularly perturbed Hamiltonian systems'*
- They consider Hamiltonian $H(x, y, u, v; \varepsilon)$ with symplectic form $d\Omega = dx \wedge dy + \varepsilon du \wedge dv$, that is

$$\varepsilon \frac{dx}{dt} = \frac{\partial H}{\partial y} \qquad \frac{du}{dt} = \frac{\partial H}{\partial v}$$
$$\varepsilon \frac{dy}{dt} = -\frac{\partial H}{\partial x} \qquad \frac{dv}{dt} = -\frac{\partial H}{\partial u}$$

- x and y are the fast variables, u and v are the slow variables
- The harmonic oscillators are in $1 : n$ resonance as $\varepsilon \to 0$
- They prove the existence of invariant tori near the equilibrium point
- They also show via numerical computations that the long period family has a series of gaps
- Here ω_1 would be the fast frequency and ω_2 the slow frequency

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Versal Normal Form for μ near 0

- The paper is not applicable when $\mu \to 0$
- Need to consider versal normal form and not diagonal form as in the paper
- $$H = \frac{\omega_1}{2}(x_1^2 + y_1^2) - \frac{1}{2}(x_2^2 + \omega_2^2 y_2^2)$$

  and

  $$\dot{z} = \begin{bmatrix} 0 & 0 & \omega_1 & 0 \\ 0 & 0 & 0 & -\omega_2^2 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} z$$

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Symplectic linear transformation with

$$R = \begin{bmatrix} \gamma & 1/4+\omega_2^2 & 8\omega_1 & -\gamma/4 \\ -7+4\omega_2^2 & -\gamma/4 & 0 & 3/4-\omega_2^2 \\ -1+4\omega_2^2 & 0 & \gamma\omega_1 & (-3+3\omega_2^2-4\omega_2^4)/4 \\ \gamma & 1 & \omega_1+4\omega_1\omega_2^2 & \gamma(-1+\omega_2^2)/4 \end{bmatrix}$$

$$S = \begin{bmatrix} \frac{1}{2\sqrt{\omega_1(1-2\omega_2^2)(7-4\omega_2^2)}} & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{(1-2\omega_2^2)(3-3\omega_2^2+4\omega_2^4)}} & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{\omega_1(1-2\omega_2^2)(7-4\omega_2^2)}} & 0 \\ 0 & 0 & 0 & \frac{2}{\sqrt{(1-2\omega_2^2)(3-3\omega_2^2+4\omega_2^4)}} \end{bmatrix}$$

Transformation is regular for $\mu = 0$

- Easy to check, set $\omega_2 = 0$
- ω_2 is a natural parameter for the problem
- Replacement rules

$$\begin{aligned} \omega_1^2 &\rightarrow 1 - \omega_2^2 \\ \gamma^2 &\rightarrow 27 - 16\omega_2^2 + 16\omega_2^4 \end{aligned}$$

Normalization of higher order terms via Lie transform

$$H = H_0^0(x_1, x_2, y_1, y_2) + H_1^0(x_1, x_2, y_1, y_2) + \frac{1}{2!}H_2^0(x_1, x_2, y_1, y_2) + \cdots$$

with

$$H_0^0 = \frac{\omega_1}{2}(x_1^2 + y_1^2) - \frac{1}{2}(x_2^2 + \omega_2^2 y_2^2)$$

Normalization is carried out in real variables, to avoid any issues with reality conditions

For

$$W = x_1^{\alpha_1} x_2^{\alpha_2} y_1^{\beta_1} y_2^{\beta_2}$$

we have

$$\begin{aligned} L_W H_0^0 &= \beta_1\omega_1 x_1^{\alpha_1+1} x_2^{\alpha_2} y_1^{\beta_1-1} y_2^{\beta_2} - \alpha_1\omega_1 x_1^{\alpha_1-1} x_2^{\alpha_2} y_1^{\beta_1+1} y_2^{\beta_2} \\ &\quad -\beta_2 x_1^{\alpha_1} x_2^{\alpha_2+1} y_1^{\beta_1} y_2^{\beta_2-1} + \alpha_2\omega_2^2 x_1^{\alpha_1} x_2^{\alpha_2-1} y_1^{\beta_1} y_2^{\beta_2+1} \end{aligned}$$

Invariant subspace for terms of degree

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = n$$

is formed by $\alpha_1 + \beta_1 = n_1$ and $\alpha_2 + \beta_2 = n - n_1$

Action of $L_W H_0^0$ in matrix form
For invariant sub-spaces $(y_1^3, y_1^2 x_1, y_1 x_1^2, x_1^3)$

$$\begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -3\omega_1 & 0 & 2\omega_1 & 0 \\ 0 & -2\omega_1 & 0 & 3\omega_1 \\ 0 & 0 & -\omega_1 & 0 \end{pmatrix}$$

Matrix is nonsingular and thus all third order terms in this sub-space can be eliminated

For invariant sub-spaces $(y_1^4, y_1^3 x_1, y_1^2 x_1^2, y_1 x_1^3, x_1^4)$

$$\begin{pmatrix} 0 & \omega_1 & 0 & 0 & 0 \\ -4\omega_1 & 0 & 2\omega_1 & 0 & 0 \\ 0 & -3\omega_1 & 0 & 3\omega_1 & 0 \\ 0 & 0 & -2\omega_1 & 0 & 4\omega_1 \\ 0 & 0 & 0 & -\omega_1 & 0 \end{pmatrix}$$

Matrix is singular and thus not all fourth order terms in this sub-space can be eliminated. Common to choose terms in kernel of form $(x_1^2 + y_1^2)^2$, that is, terms make up action variable

$$I_1 = \frac{1}{2}(x_1^2 + y_1^2)$$

Action of $L_W H_0^0$ in matrix form
For invariant sub-spaces $(y_2^3, y_2^2 x_2, y_2 x_2^2, x_2^3)$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 3\omega_2^2 & 0 & -2 & 0 \\ 0 & 2\omega_2^2 & 0 & -3 \\ 0 & 0 & \omega_2^2 & 0 \end{pmatrix}$$

- Matrix is singular when $\omega_2 = 0$
- Can not eliminate all third order terms in this sub-space
- Will keep term with y_2^3
- Same will happen at fourth order terms
- Will keep terms with y_2^4

Full versal normal form at $\mathcal{L}_4$

$$\begin{aligned}\tilde{H} &= \omega_1 I_1 - \frac{1}{2}(x_2^2 + \omega_2^2 y_2^2) \\ &\quad +\omega_2^2(a_1 I_1 y_2 + a_2 y_2^3 \\ &\quad +b_1 I_1^2 + b_2 I_1 y_2^2 + b_3 y_2^4 \\ &\quad +c_1 I_1^2 y_2 + c_2 I_1 y_2^3 + c_3 y_2^5 \\ &\quad +\cdots) \\ \\ &= \omega_1 I_1 - \frac{1}{2}(x_2^2 + \omega_2^2 y_2^2) + F(I_1, y_2)\end{aligned}$$

All coefficients depend on ω_2 and are continuous at $\omega_2 = 0$

Differential Equations for Versal Normal Form

$$\tilde{H} = \omega_1 I_1 - \frac{1}{2}(x_2^2 + \omega_2^2 y_2^2) + F(I_1, y_2)$$

$$\begin{aligned}
\dot{I}_1 &= 0 \\
\dot{\phi}_1 &= -\omega_1 - \frac{\partial F}{\partial I_1} \\
\dot{x}_2 &= -\omega_2^2 y_2 + \frac{\partial F}{\partial y_2} \\
\dot{y}_2 &= x_2
\end{aligned}$$

Short Period Family

- Since $\frac{\partial F}{\partial y_2} \neq 0$ for $x_2 = y_2 = 0$ short period family no longer at $x_2 = y_2 = 0$ but nearby
- Also period will not be exactly $2\pi/\omega_1$.
- The equation

$$\ddot{y}_2 = -\omega_2^2 y_2 + \frac{\partial F}{\partial y_2}$$

has a $2\pi/\omega_1$ periodic (that is constant) solution if

$$y_2 = \frac{1}{\omega_2^2} \frac{\partial F}{\partial y_2}$$

and with it $x_2 = 0$

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Short Period Family Continued: I_1=const

Set

$$y_2 = 0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots$$

we find the constant solution

$$y_2 = \varepsilon a_1 I_1 + \varepsilon^3(3a_1^2 + 2a_1 b_2 + c_1)I_1^2 + \cdots$$

and

$$\dot\phi_1 = \omega_1 + \varepsilon(a_1^2 + b_1)\omega_2^2 I_1 + \varepsilon^3 3(a_1^3 a_2 + a_1^2 b_2 + a_1 c_1 + d_1)\omega_2^2 I_1^2 + \cdots$$

Remarks

- Lie Transformation has been carried out to a much higher order
- Choice of ω_2 as the basic parameter makes this possible
- For restricted three body problem at $\mathcal{L}_4$ rational expressions in ω_2 are generated.
- At each step of the normalization new singularities are created, they are the same as appear when normalizing $H = \omega_1 I_1 - \omega_2 I_2$

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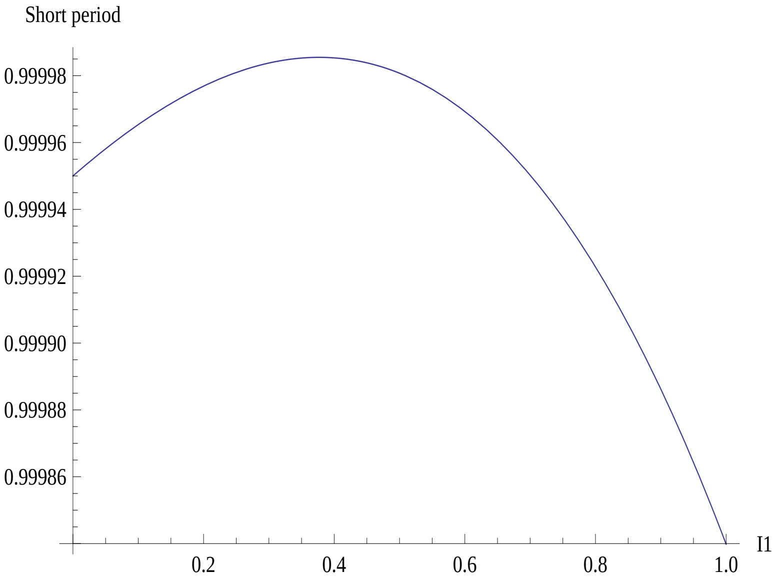
Frequency: Short Period Family

$$\sqrt{1-\omega_2^2}+\omega_2^2\times$$
$$\Big(\frac{I_1(-491+448\omega_2^2+124\omega_2^4)}{72(1-2\omega_2^2)^2(-4+5\omega_2^2)}$$
$$-\frac{I_1^2 p_2(\omega_2)}{20736\sqrt{1-\omega_2^2}(-1+2\omega_2^2)^5(-4+5\omega_2^2)^3(-9+10\omega_2^2)}$$
$$-\frac{I_1^3 p_3(\omega_2)}{13436928(9-10\omega_2^2)^2(1-2\omega_2^2)^8(-4+5\omega_2^2)^5(16-33\omega_2^2+17\omega_2^4)}$$
$$+\cdots\Big)$$

$$\begin{aligned} p_2(\omega_2) \quad = \quad & (-18522432-221117724\omega_2^2+1834402891\omega_2^4-5330237408\omega_2^6+8326473644\omega_2^8 \\ & -7970990576\omega_2^{10}+4915656752\omega_2^{12}-1885370432\omega_2^{14}+349789120\omega_2^{16}) \end{aligned}$$

Frequency of short period orbits for $\omega_2 = 0.01$



Arnold's Stability Theorem for a Hamiltonian with two degrees of freedom

- Arnold's theorem addresses the case when exponents are pure imaginary, and the Hamiltonian is not positive definite.
- Assume the Hamiltonian has been normalized, that is in symplectic coordinates x_1, x_2, y_1, y_2 of the form

$$H = H_2 + H_4 + \cdots + H_{2N} + H^\dagger,$$

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Arnold's Stability Theorem

- $H = H_2 + H_4 + \cdots + H_{2N} + H^\dagger$,
- H_{2k}, $1 \le k \le N$, is a homogeneous polynomial of degree k in I_1, I_2
- Series expansion of $H^\dagger$ starts with terms of degree $2N + 1$;
- $H_2 = \omega_1 I_1 - \omega_2 I_2$, ω_i nonzero constants;

Theorem

The origin is stable provided that for some k, $0 \le k \le N$, $D_{2k} = H_{2k}(\omega_2, \omega_1) \neq 0$ or, equivalently, provided H_2 does not divide H_{2k}. In particular, the equilibrium is stable if

$$D_4 = \frac{1}{2}\{A\omega_2^2 + 2B\omega_1\omega_2 + C\omega_1^2\} \neq 0.$$

Moreover, arbitrarily close to the origin in $\mathbb{R}^4$, there are invariant tori and the flow on these invariant tori is the linear flow with irrational slope.

Stability of $\mathcal{L}_4$ for $0 < \mu < \mu_1$, $\mu \neq \mu_2$ and $\mu \neq \mu_3$

- From the values of A, B and C given earlier, compute

$$D_4 = -\frac{36 - 541\omega_1^2\omega_2^2 + 644\omega_1^4\omega_2^4}{8(1 - 4\omega_1^2\omega_2^2)(4 - 25\omega_1^2\omega_2^2)},$$

- With $\omega_1^2\omega_2^2 = \frac{27}{4}\mu(1-\mu)$ solve $D_4 = 0$ and find four real roots

$$\mu = \frac{1}{2} \pm \frac{1}{6}\sqrt{(3265 \pm 2\sqrt{199945})/483}$$

- $\mu = 0.0109137$, $\mu = 0.130756$, $\mu = 0.869244$, $\mu = 0.989086$
- The first value $\mu_d = \frac{1}{2} - \frac{1}{6}\sqrt{(3265 - 2\sqrt{199945})/483}$ is in the interval $(0, \mu_1)$

The special case μ_d

- need to carry out normalization to terms of order six
- $D_6 = 4!H_0^4(\omega_2, \omega_1) = P/Q$
-

$$
\begin{aligned}
P &= -\frac{3105}{4} + \frac{1338449}{48}\sigma - \frac{489918305}{1728}\sigma^2 + \frac{7787081027}{6912}\sigma^3 \\
&\quad - \frac{2052731645}{1296}\sigma^4 - \frac{1629138643}{324}\sigma^5 \\
&\quad + \frac{1879982900}{81}\sigma^6 + \frac{368284375}{81}\sigma^7, \\
Q &= \omega_1\omega_2(\omega_1^2 - \omega_2^2)^5(4 - 25\sigma)^3(9 - 100\sigma), \\
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Remarks

- Solve $D_4 = 0$ for σ and substitute into D_6 to get $D_6 \approx -66.6$
- Thus $\mathcal{L}_4$ is stable at μ_d
- $\mu_d = 0.010913667677201$ does not appear to have any specific significance
- The significance of $\mu^* = 0.012723988746541 63$ is also not clear, except that the structure of the family with common period changes from the "open" case to "bridges"
- $\mu_4 < \mu_d < \mu^* < \mu_3$. It would have been nice if $\mu_d = \mu^*$
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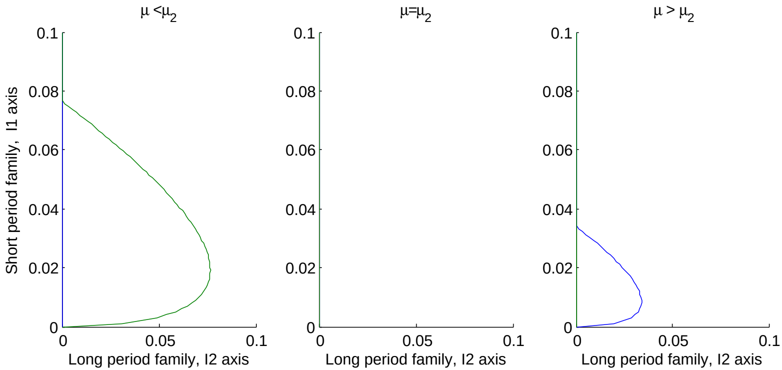
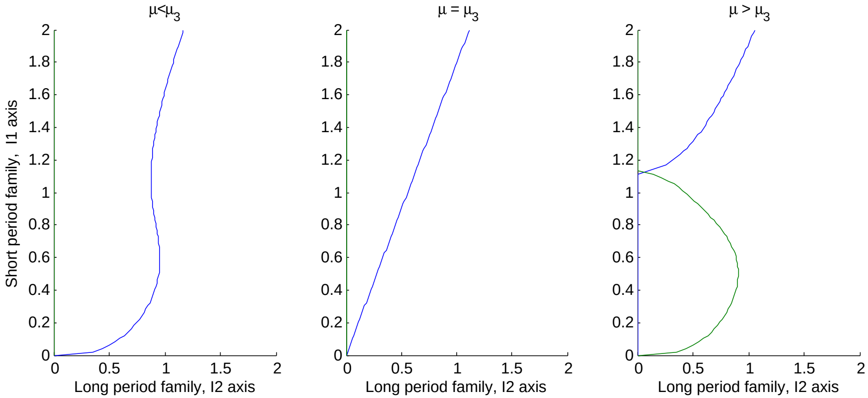
Remarks to μ near μ_2



Remarks to μ near μ_3



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