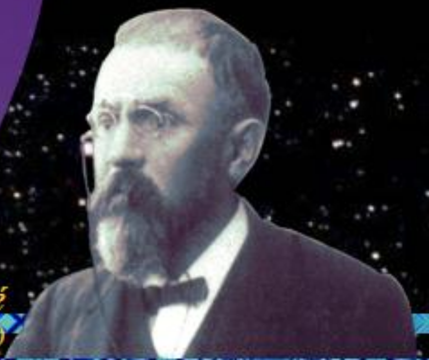


*Jules Henri Poincaré*
(1854 – 1912)

# XVI Colóquio Brasileiro de Dinâmica Orbital

**Hotel Biazi, Serra Negra/SP**
**26 a 30 de novembro/2012**

Universidade Federal do ABC

Associação Aeroespacial Brasileira

CNPq Conselho Nacional de Desenvolvimento Científico e Tecnológico

FAPESP Fundação de Amparo à Pesquisa do Estado de São Paulo

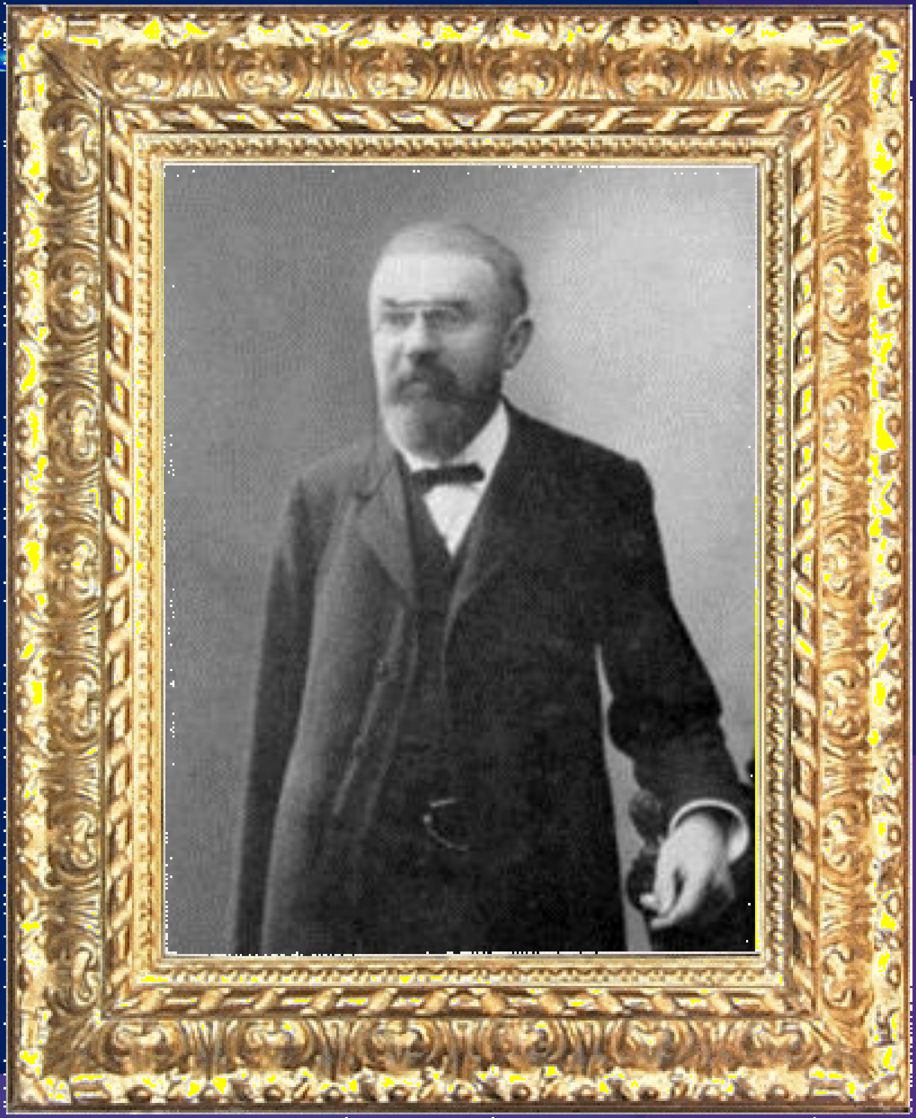
CAPES


**Annibal Hetem**
Universidade Federal do ABC, UFABC, Santo André, SP, Brasil
annibal.hetem@ufabc.edu.br

# POINCARÉ AND THE CELESTIAL MATHEMATICS: A CENTURY OF FINE CONJECTURES

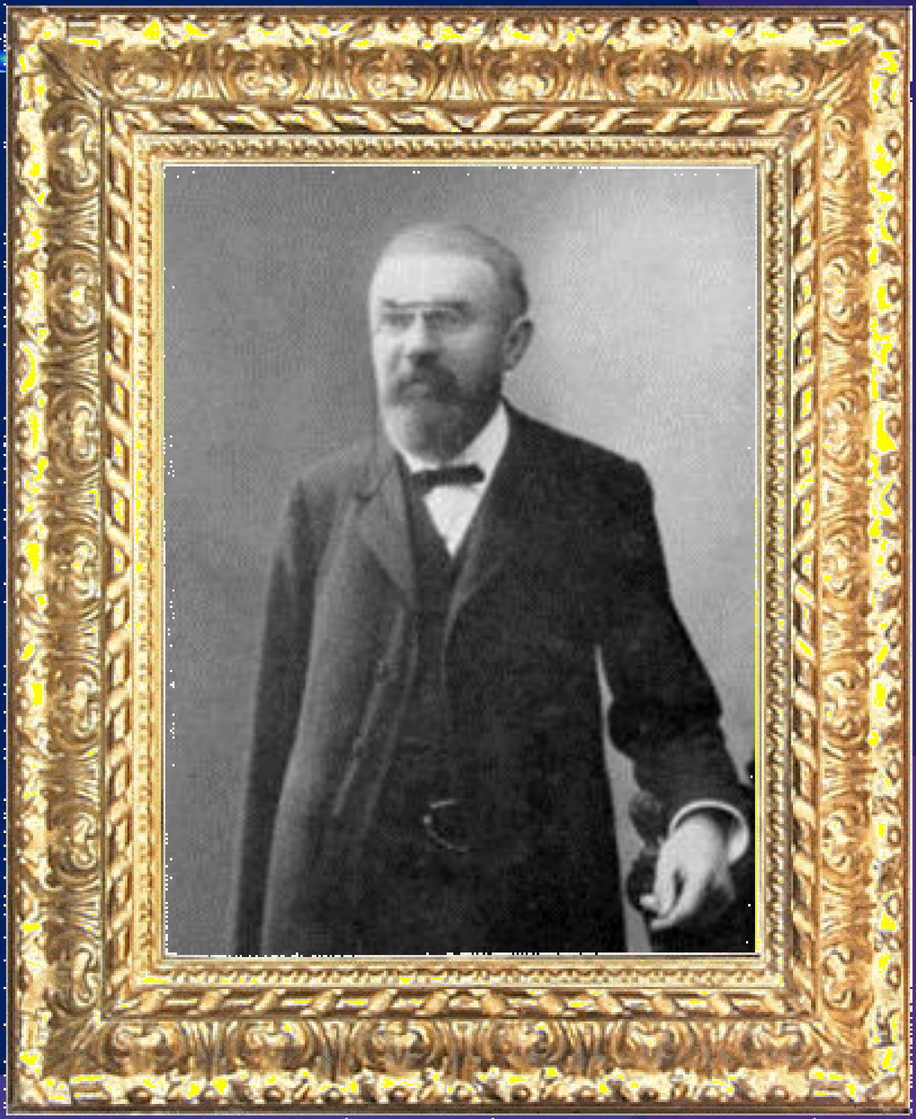
# Jules Henri Poincaré

- Mathematician
- Theoretical physicist
- Engineer (mines)
- Philosopher of science



# Awards

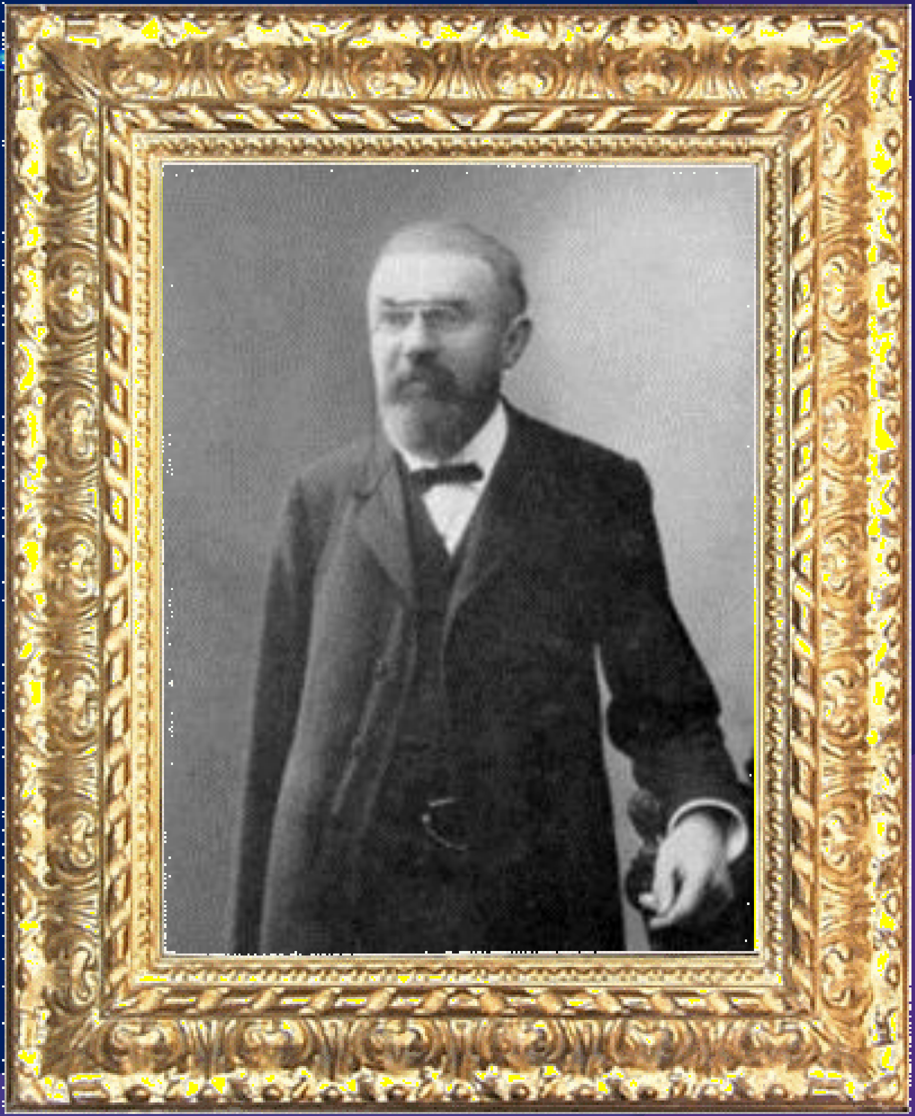
- Poncelet Prize (1885)
- Oscar II, King of Sweden's mathematical competition (1887)
- American Philosophical Society (1899)
- Gold Medal of the Royal Astronomical Society of London (1900)
- Sylvester Medal (1901)
- Bolyai Prize (1905)
- Matteucci Medal (1905)
- French Academy of Sciences (1906)
- Académie Française (1909)
- Bruce Medal (1911)

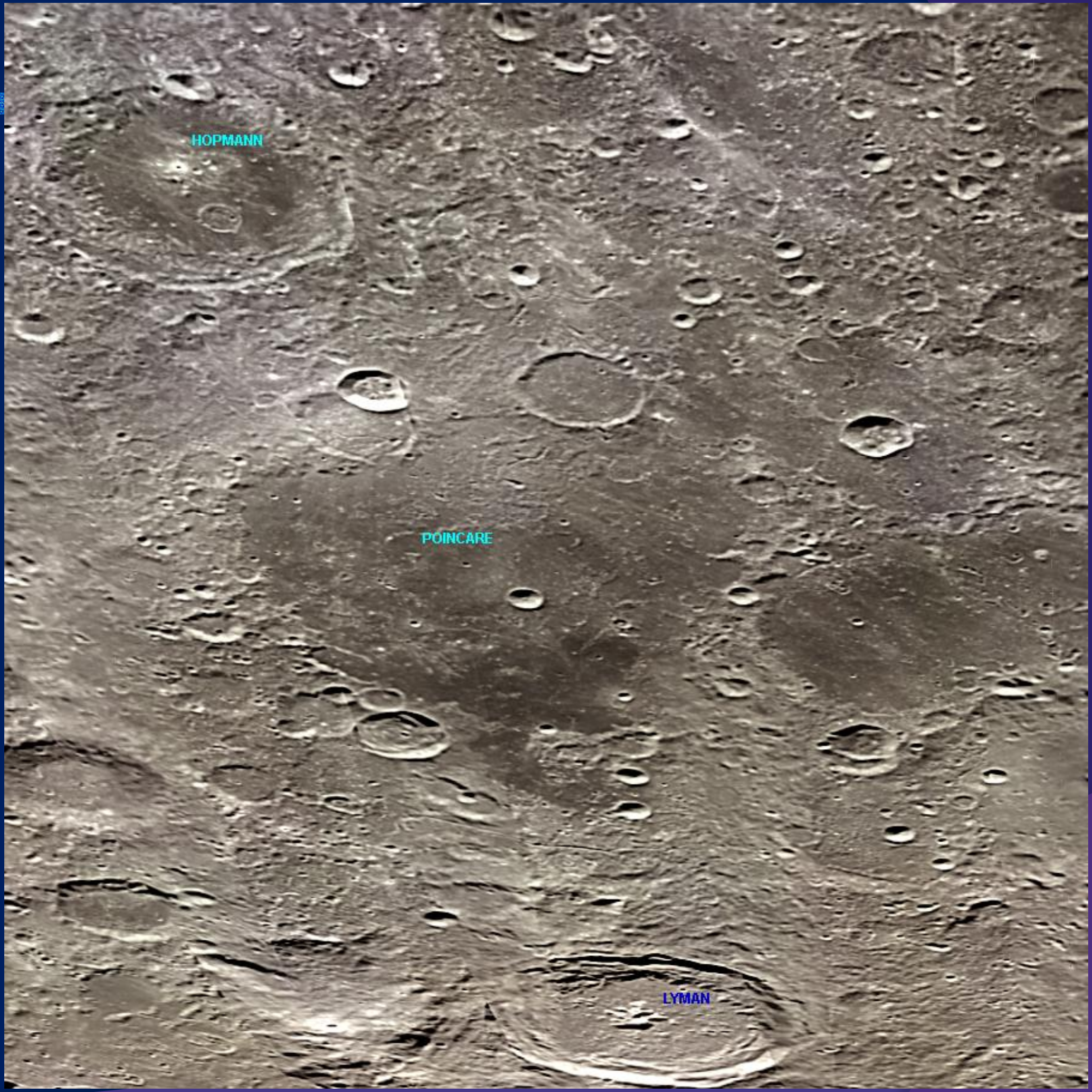


(there are some more...)

# Homages

- **Institut Henri Poincaré** (mathematics and theoretical physics center)
- **Poincaré Prize** (Mathematical Physics International Prize)
- **Annales Henri Poincaré** (Scientific Journal)
- **Poincaré Seminar** (the "Bourbaphy")
- **Asteroid** 2021 Poincaré
- **The crater Poincaré** on the Moon






Poincaré crater in the Moon, 319 km diameter

# Influential philosopher of science and mathematics

- In the foundations of mathematics he argued for conventionalism against formalism and logicism.
- He stressed the essential role of intuition in a proper constructive foundation for mathematics.
- He believed mathematicians can use the methods of logic to check a proof, but they must use intuition to create a proof.

# *“The aim of science is prediction rather than explanation”*

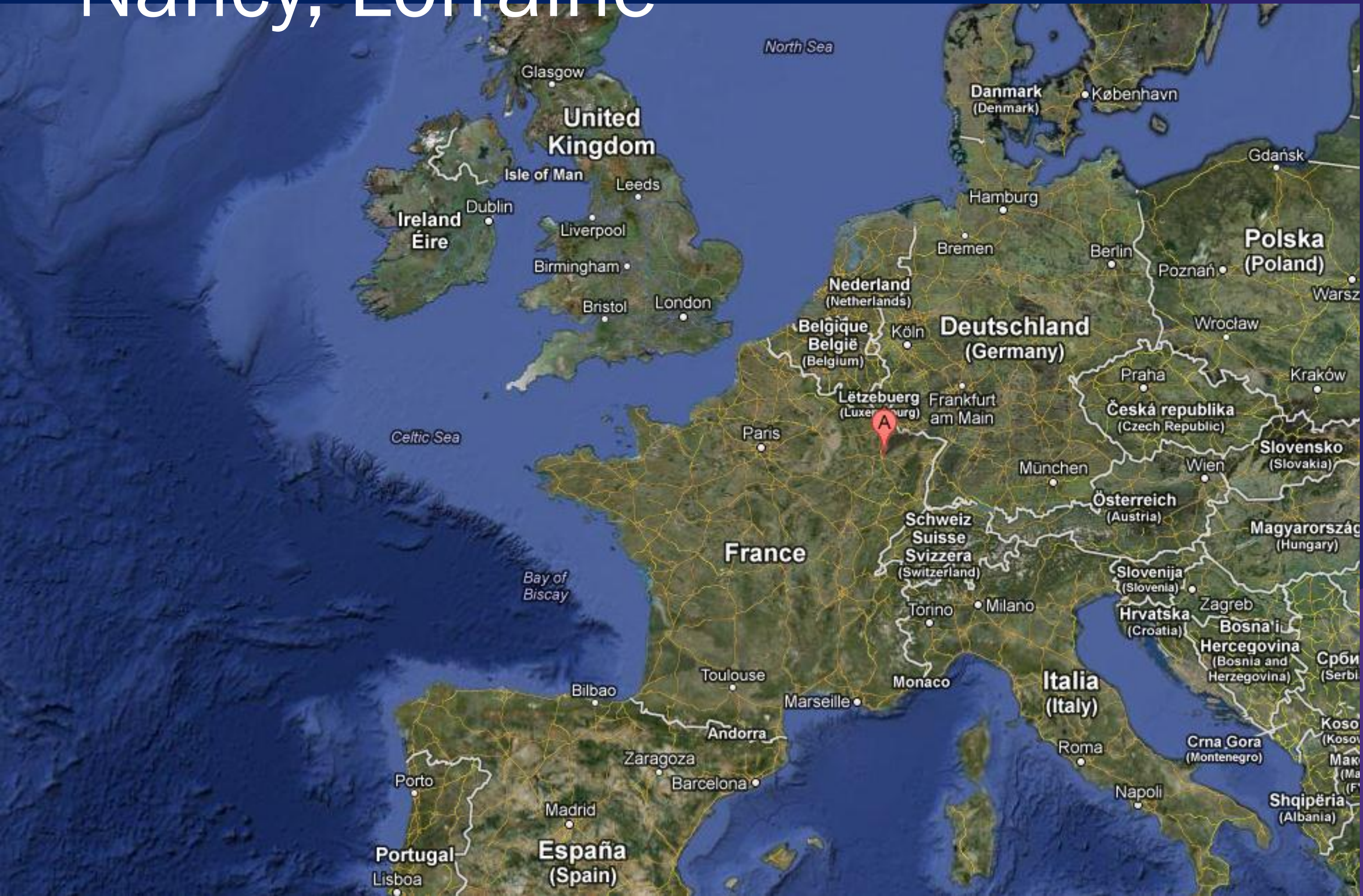
- Poincaré believed that scientific laws are conventions but not arbitrary conventions.

# 1. Life

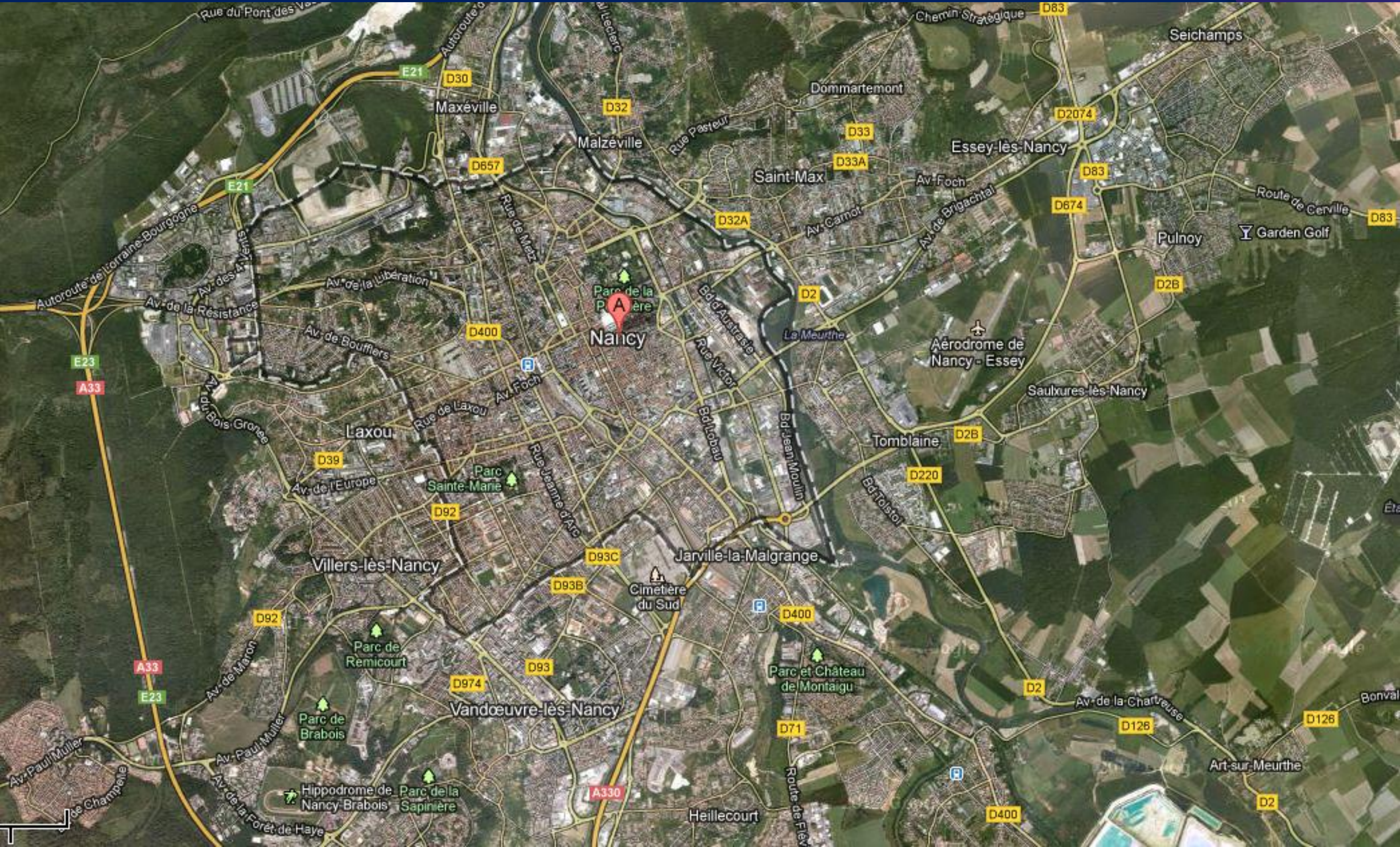
*29 April 1854 Nancy, Meurthe-et-Moselle, France*

*17 July 1912 Paris, France*

# Nancy, Lorraine



# Nancy, Lorraine

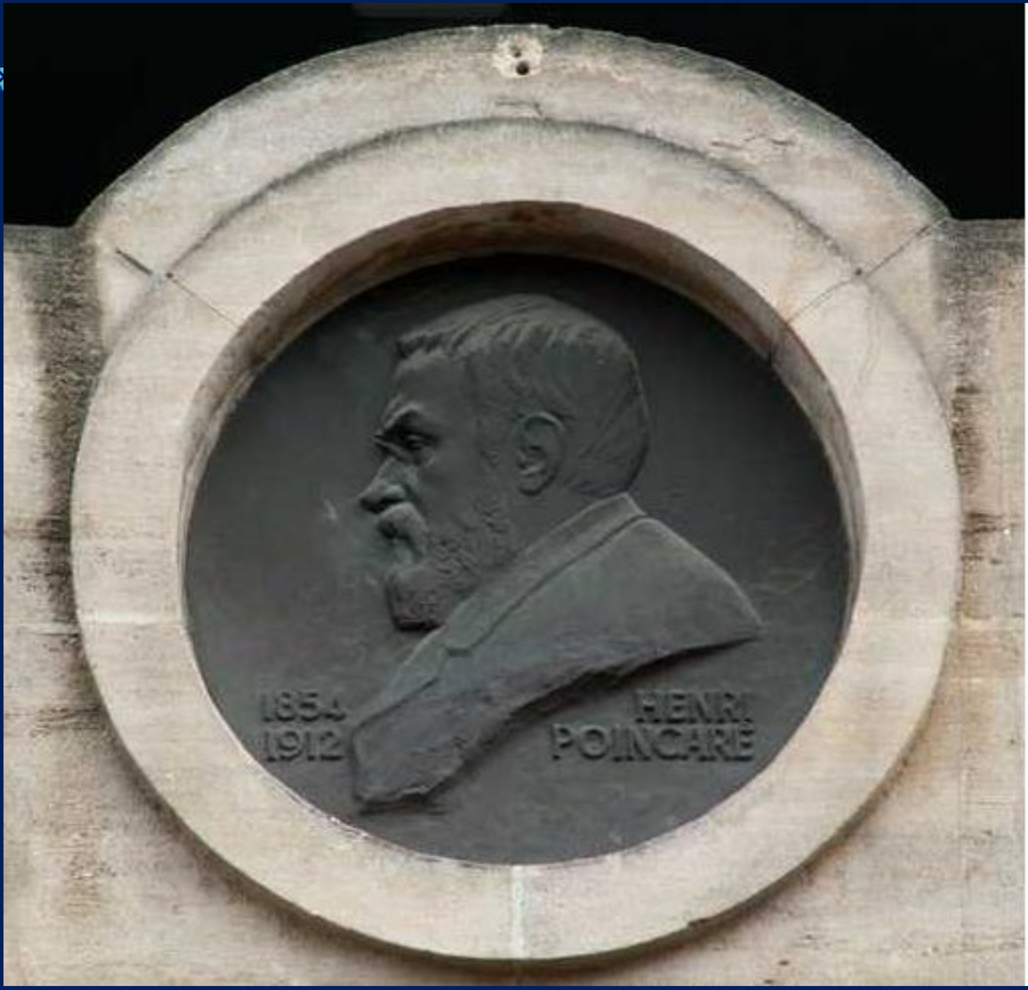


# Nancy, Lorraine

Département: Meurthe et Moselle
Population: 103 552 hab.
Superficie: 15,01 km²
Code postal: 54000

Porte De La Graffe, Old Town.

# Lycée Henri Poincaré




Rue Henri Poincaré 41
Nancy, Meurthe-et-Moselle / Lorraine

# Family

- Poincaré's family was influential.
- His father was a professor of medicine at the University of Nancy.
- His sister Aline married the spiritualist philosopher Emile Boutroux.

# Cousin Raymond

- **Raymond Poincaré**

  President of France
  (1913-1920)

Raymond Poincaré (1913)

# Childhood

- Poincaré was known for being polite, distracted and ambidextrous.
- He had a photographic memory and never took notes.
- His homework´s were written in crumpled paper that he took from his pocket.

Aimé Morot's
*La bataille de Reichshoffen* (1887)

# The Franco-Prussian War

According to Darboux
(President of the French Academy)
Poincaré was very shaken by the occupation and destruction of family property.

- He helped his father to treat the wounded during the war.
- Between 1870 and 1873, Prussians officials were lodged in his house.
- Poincaré used the opportunity to learn German.

# Doctoral dissertation (1879)

- ***Sur les propriétés des fonctions définies par les équations différencieles***

  under the supervision of Charles Hermite.

- Later, Poincaré applied to celestial mechanics the methods he had introduced in his doctoral dissertation.

- The methods he invented gave rise to algebraic topology.

Poincaré (1880)

# University of Paris

- Poincaré was initially appointed as the *maître de conférences d'analyse* (1881).
- Later, he held the chairs of
  - Physical and Experimental Mechanics
  - Mathematical Physics
  - Theory of Probability
  - Celestial Mechanics and Astronomy.

# Mail with Felix Klein

- During 1881, Poincaré maintained an intense dispute by correspondence with Felix Klein.
- According to Donald O'Shea, it was a "street fight with knives hidden.”
- In the fall of 1882, Klein gives up:
  *"My work in mathematics ceased production in 1882."*

Christian Felix Klein
(1849 –1925)

- *Group theory*
- *Complex analysis*
- *Non-Euclidean geometry*
- *Connections between geometry and group theory.*

# French Academy of Sciences

Poincaré (1886)

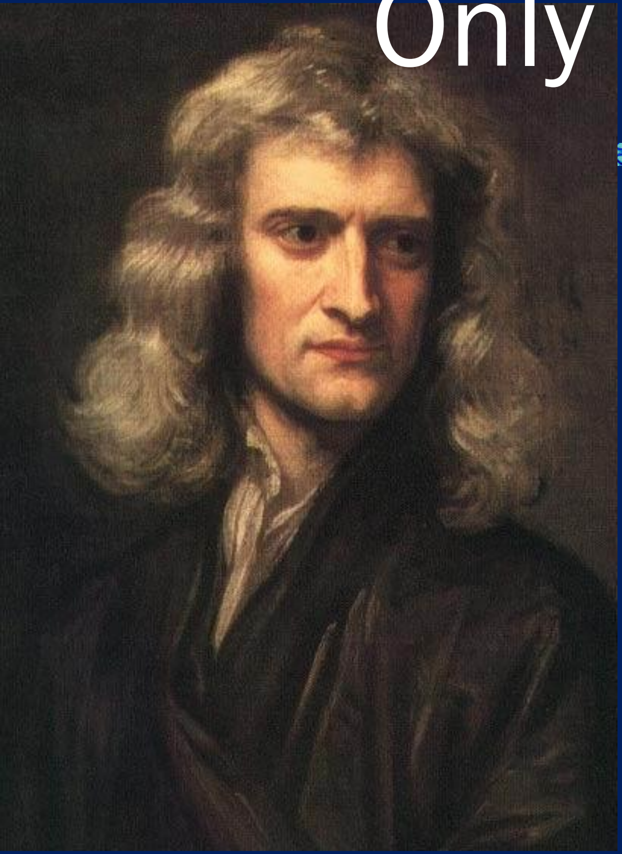
- In 1887, Poincaré was elected to the French Academy of Sciences.
  - Also the year his first son was born.
- He became its president in 1906, and was elected to the Académie française in 1909.

# Solvay Scientific Meetings


Poincaré and Mme Curie

# 2. The three body problem

# Only three bodies in space...



Sir Isaac Newton
*Philosophiae Naturalis Principia Mathematica* (1687)

Given the law of gravity and the initial positions and velocities of the only three bodies in all of space, the subsequent positions and velocities are fixed for all time.

The three-body system appears to be deterministic.

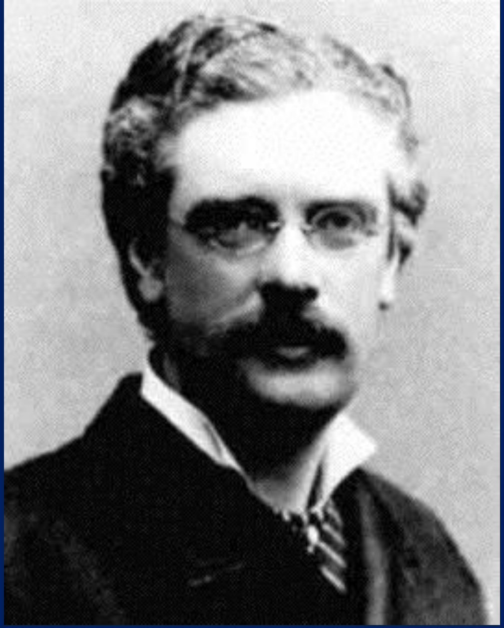
# King Oscar II of Sweden

- Mathematician, writer and musical amateur.
- Encouraged the development of education.
- Provided financial support for the founding of Acta Mathematica.

- For his 60th birthday, a mathematics competition was to be held.
- Was to be judged by an international jury of leading mathematicians.
- Offered a prize for the solution for the general *n*-body problem.

King Oscar II
(1829 – 1907)

# Jury

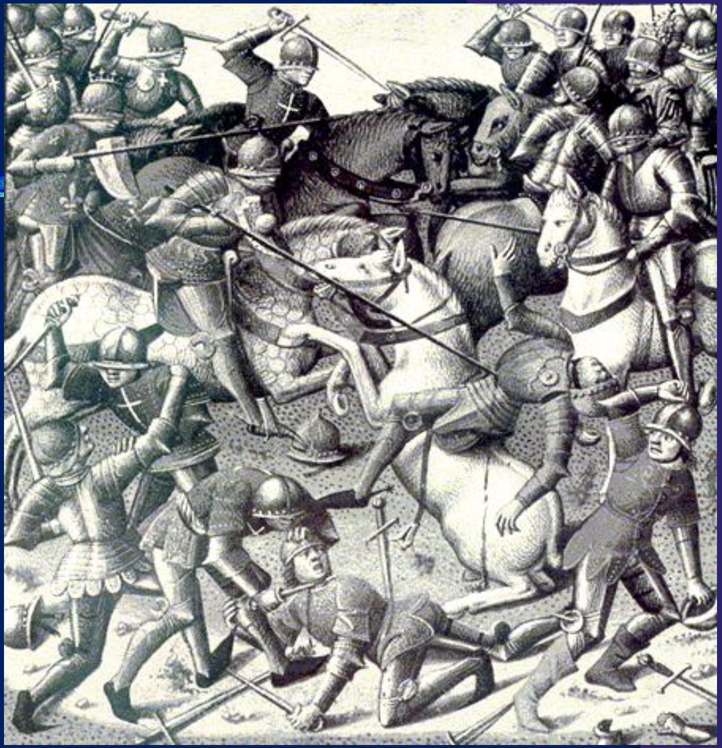

Leffler


Weierstrass


Hermite

- This jury represented each part of the mathematical world.

# The Contestants



Poincaré

- Chose the 3 body problem.
- Student of Hermite.

Paul Appell

- Professor of Rational Mechanics at Sorbonne.
- Student of Hermite.
- Chose his own topic.

Guy de Longchamps

- Complained to Hermite because he did not win.

Jean Escary

- Professor at the military school of La Fléche.

Cyrus Legg

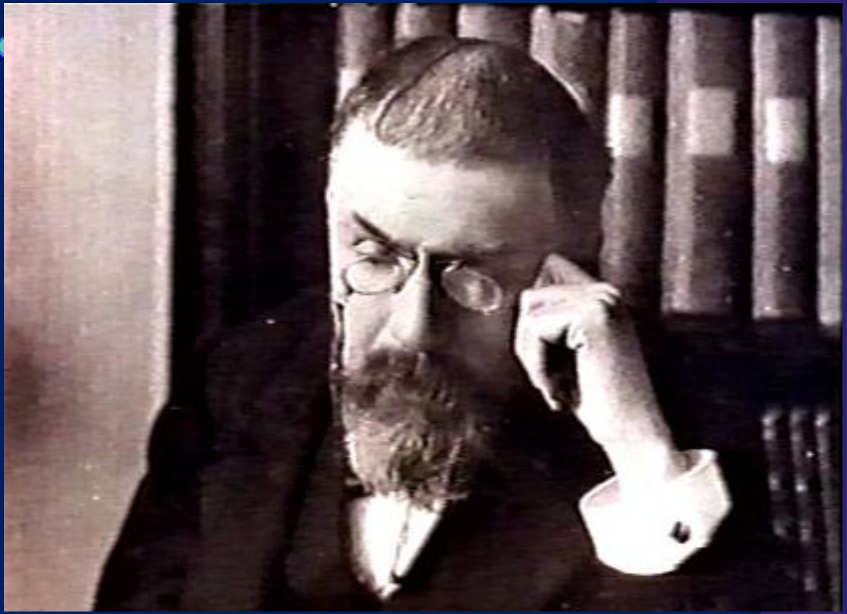
- Part of a “band of indefatigable angle trisectors”.

# Poincaré wins!

- He was unanimously chosen by the jury.
- His paper consisted of 198 pages.
- The importance of his work was obvious.
- The jury had a difficult time understanding his mathematics

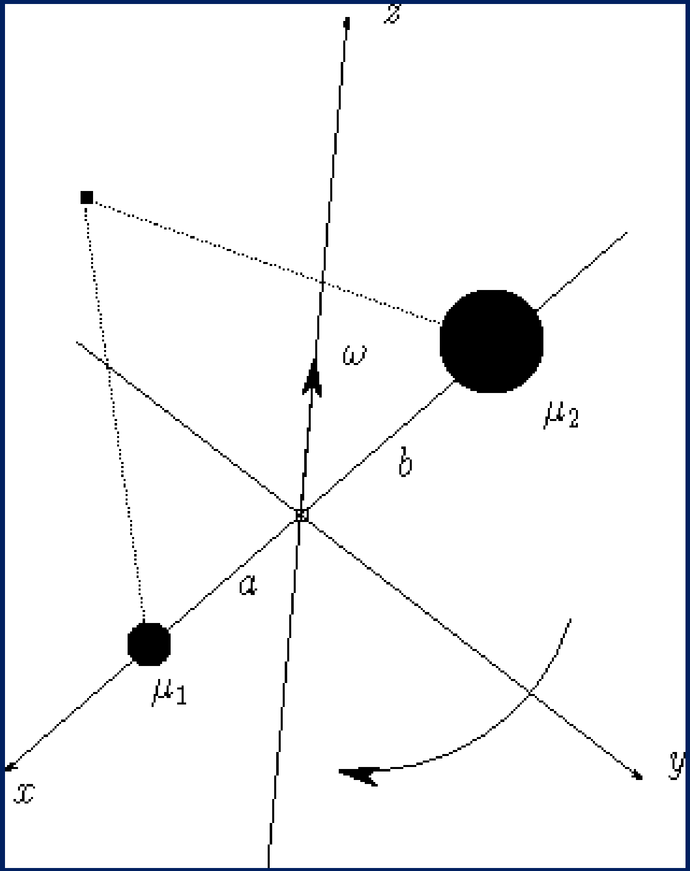
# The philosophy

*"Pénétrer dans une place jusq'ici réputée inabordable…"*



*"It's a classic three body problem, it can't be solved. It can, however, be approximated."*

# The Problem




*"I consider three masses, the first very large, the second small, but finite, and the third infinitely small: I assume that the first two describe a circle around the common center of gravity, and the third moves in the plane of the circles."*

*"An example would be the case of a small planet perturbed by Jupiter if the eccentricity of Jupiter and the inclination of the orbits are disregarded."*

-Poincaré

# The restricted 3 body problem

- Poincaré produced a solution to a modification of a generalized *n* body problem known today as The restricted 3 body problem.
- The solution has immediate application insofar as the stability of the solar system.

**"Poincaré and the Three-body Problem"**
Alain Chenciner
*Observatoire de Paris*

# 198 pages in 6 steps

1) Definitions:

$P_i$ represents the three particles

$m_i$ represents the mass of each

Distance $P_iP_j = r_{ij}$

$i=1,2,3$

The equations of motion

– Based on Newton's law of gravitation

$$\frac{d^2q_{1i}}{dt^2} = k^2 m_2 \frac{(q_{2i}-q_{1i})}{r_{12}^3} + k^2 m_3 \frac{(q_{3i}-q_{1i})}{r_{13}^3}$$

$$\frac{d^2q_{2i}}{dt^2} = k^2 m_2 \frac{(q_{1i}-q_{2i})}{r_{12}^3} + k^2 m_3 \frac{(q_{3i}-q_{2i})}{r_{23}^3}$$

$$\frac{d^2q_{3i}}{dt^2} = k^2 m_2 \frac{(q_{1i}-q_{3i})}{r_{13}^3} + k^2 m_3 \frac{(q_{2i}-q_{3i})}{r_{23}^3}$$

2) We need to reduce the order of the system of equations

Choose $k^2=1$

Force between $i$ and $j$ becomes: $-\frac{m_i m_j}{r_{ij}^2}$

Potential energy of the entire system

$$V = -\frac{m_2 m_3}{r_{23}} - \frac{m_3 m_1}{r_{31}} - \frac{m_1 m_2}{r_{12}}$$

$$p_{ij} = m_i \frac{dq_{ij}}{dt} \qquad H = \sum_{i,j=1}^{3} \frac{p_{ij}^2}{2m_i} + V$$

3) Equations in the Hamiltonian form:

$$\frac{dq_{ij}}{dt} = -\frac{\partial H}{\partial p_{ij}} \qquad \frac{dp_{ij}}{dt} = -\frac{\partial H}{\partial q_{ij}}$$

We now have a set of 18 first order differential equations.

To reduce them, we start by multiply original equations of motion by

$$m_i\left[\frac{d^2q_{ij}}{dt^2}\right] \rightarrow \sum_{i=1}^{3} m_i \frac{d^2q_{ij}}{dt^2} = 0$$

Integrate twice

$$\iint dt \sum_{i=1}^{3} m_i \frac{d^2q_{ij}}{dt^2} = \sum_{i=i}^{3} m_i q_{ij} = A_j t + B_j$$

$A_j$ and $B_j$ are constants of integration

4) Since the integral is a constant, the motion of the center of mass is either stationary or moving at constant velocity.

Multiply:

$$-q_{12}\frac{d^2q_{11}}{dt^2} \quad -q_{22}\frac{d^2q_{12}}{dt^2} \quad -q_{32}\frac{d^2q_{13}}{dt^2}$$

and

$$q_{11}\frac{d^2q_{21}}{dt^2} \quad q_{21}\frac{d^2q_{22}}{dt^2} \quad q_{31}\frac{d^2q_{23}}{dt^2}$$

Then add the two together to get

$$\sum_{i=1}^{3} m_i q_{i1} \frac{d^2q_{i2}}{dt^2} - \sum_{i=1}^{3} m_i q_{i2} \frac{d^2q_{i1}}{dt^2} = 0$$

5) Permute cyclically the variable and integrate to obtain

$$\sum_{i=1}^{3} m_i\left(q_{i2}\frac{dq_{i3}}{dt} - q_{i3}\frac{dq_{i2}}{dt}\right) = C_1$$

$$\sum_{i=1}^{3} m_i\left(q_{i3}\frac{dq_{i1}}{dt} - q_{i1}\frac{dq_{i3}}{dt}\right) = C_2$$

$$\sum_{i=1}^{3} m_i\left(q_{i1}\frac{dq_{i2}}{dt} - q_{i2}\frac{dq_{i1}}{dt}\right) = C_3$$

Consider $\frac{\partial}{\partial q_{ij}}\left(\frac{1}{r_{ik}}\right) = -\frac{q_{kj}-q_{ij}}{r_{ik}^3}$

then $m_i \frac{d^2q_{ij}}{dt^2} = -\frac{\partial V}{\partial q_{ij}}$

6) Multiply by $\frac{dq_{ij}}{dt}$ and sum to get

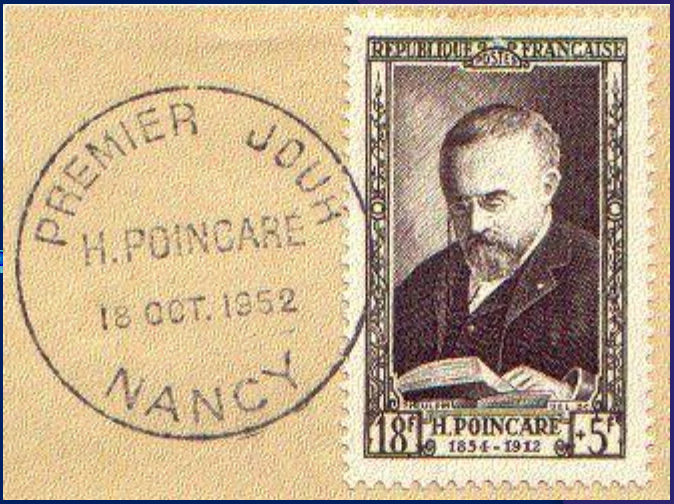
$$\sum_{i,j=1}^{3} p_{ij} \frac{d^2q_{ij}}{dt^2} = -\frac{dV}{dt}$$

Integrate: $\sum_{i,j=1}^{3} \frac{p_{ij}^2}{2m_i} = -V + C$

The final reduction is the elimination of the time variable by using a dependent variable as an independent variable.

Then a reduction through elimination of the nodes.

# 3-body Solution




Then we apply the obtained solution to the restricted three body problem.

Poincaré approach: Periodic Solutions

- His final solution to the system of differential equations:

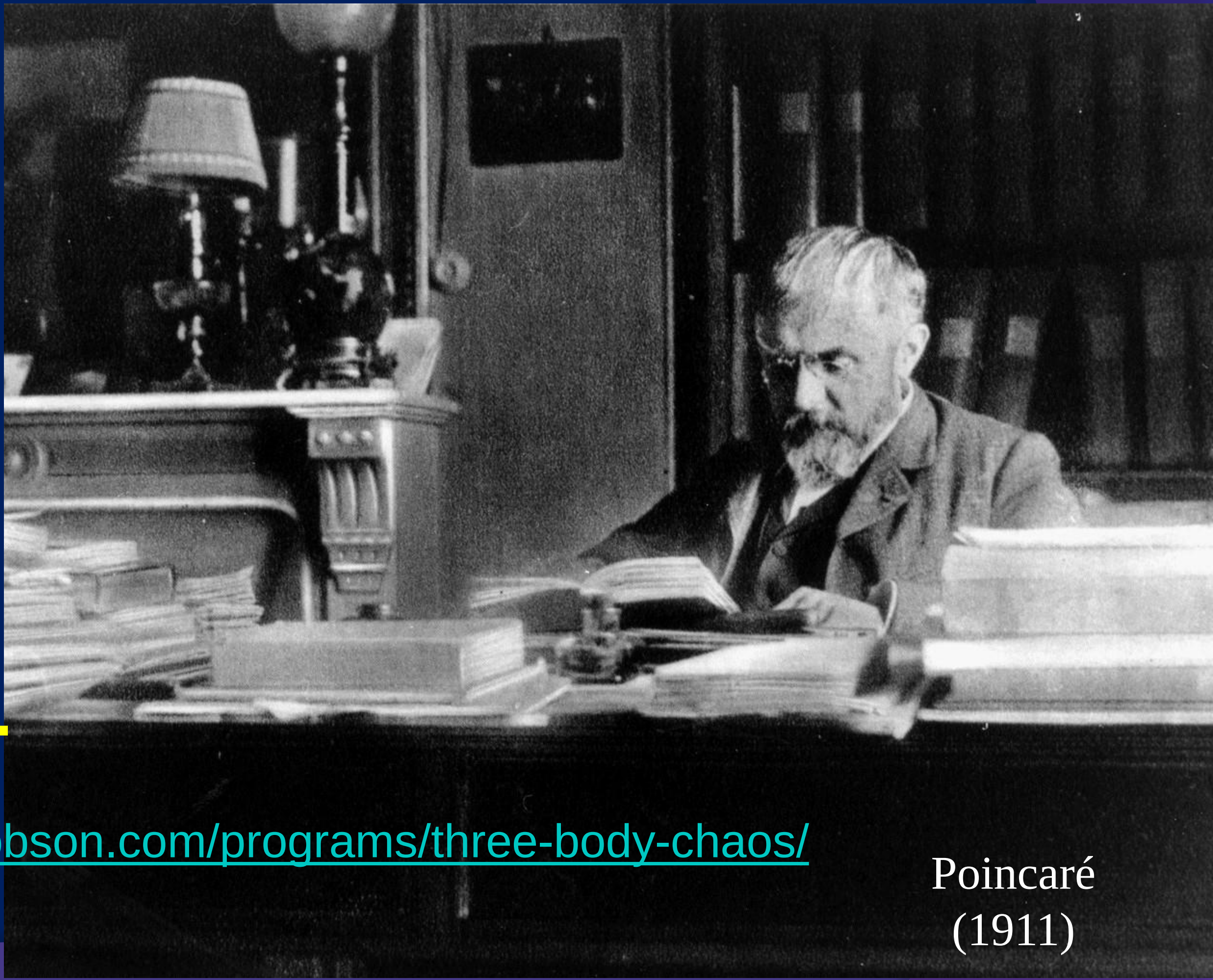
$$x_i = e^{\alpha_1 t} \lambda_{1.i}(t)$$

The rest of Poincaré's solution was an attempt to generalize the solution for the $n$ body problem

# Small changes ≠ Small Effects

The problem is harder than anyone realised.

Poincaré was forced to develop a mathematical theory of ’small bumps’, known as **asymptotic theory.**



http://alecjacobson.com/programs/three-body-chaos/

Poincaré (1911)

# A glimpse of chaos

Poincaré was the first person to discover a chaotic deterministic system during his research on the three-body problem.

The evolution of such a system is often chaotic in the sense that a small perturbation might lead to a radically different later state.

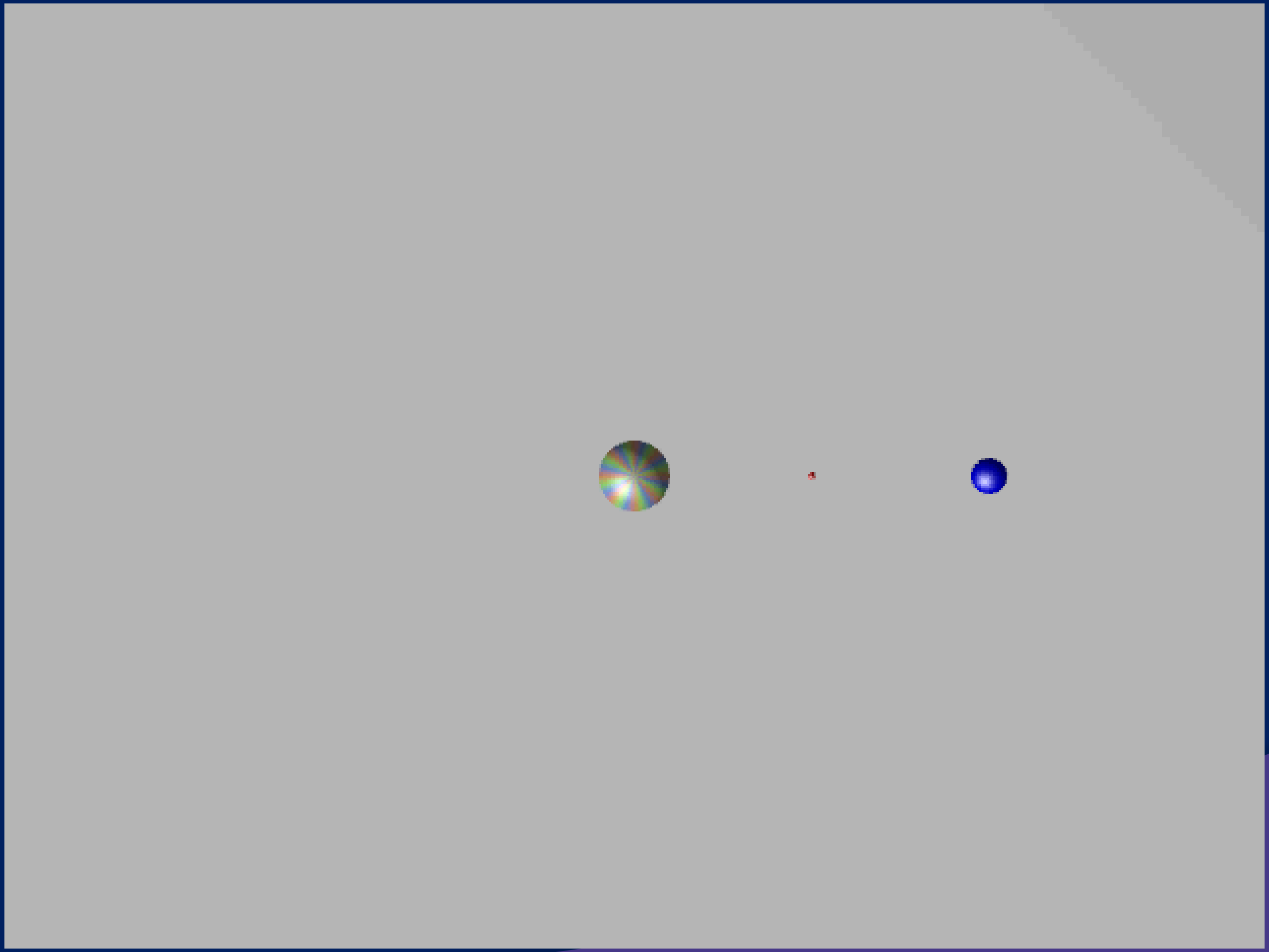
in the initial state or a slight change in one body's initial position

Poincaré in his office (1909)

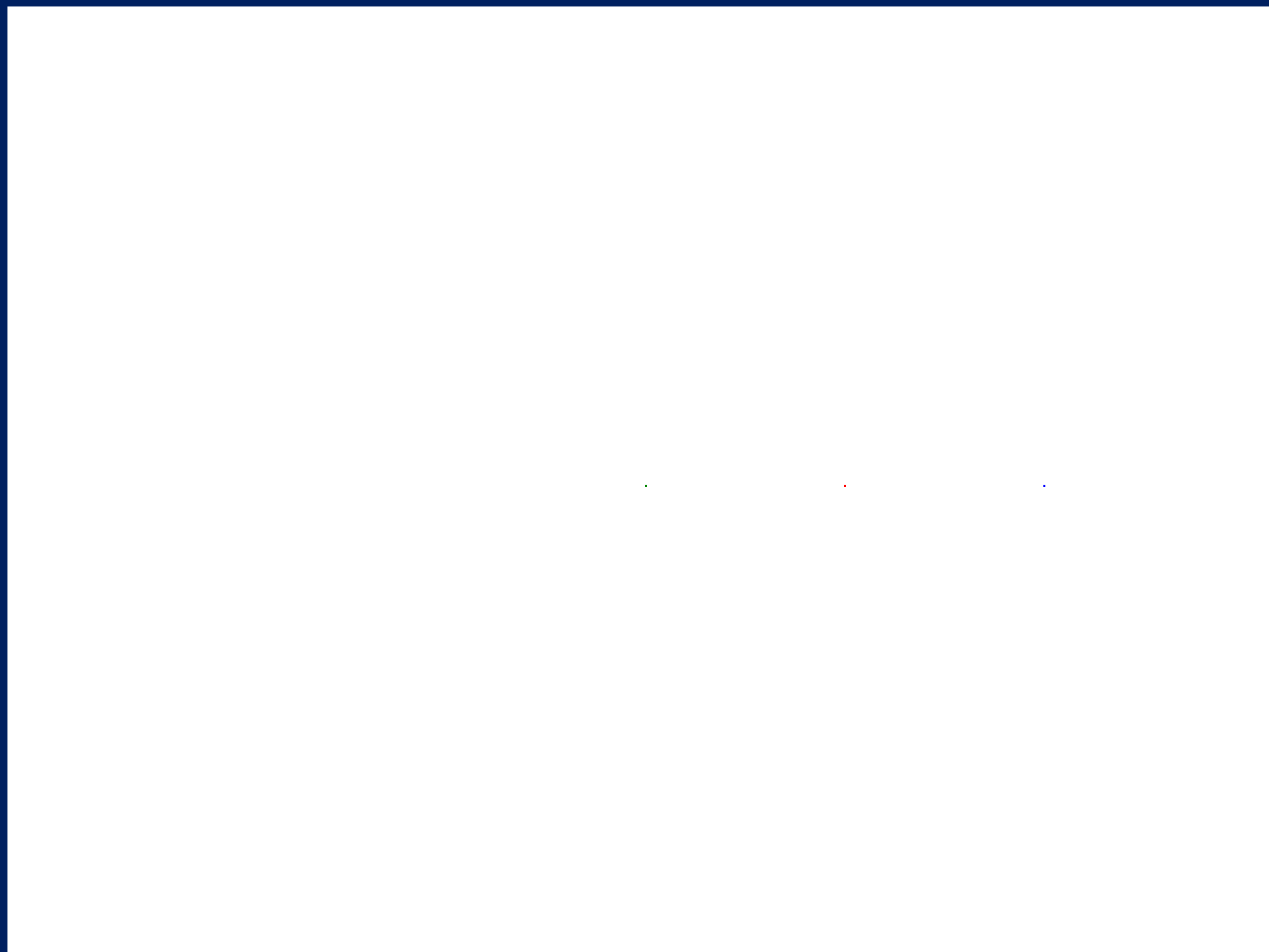
# Determinism *versus* Predictability

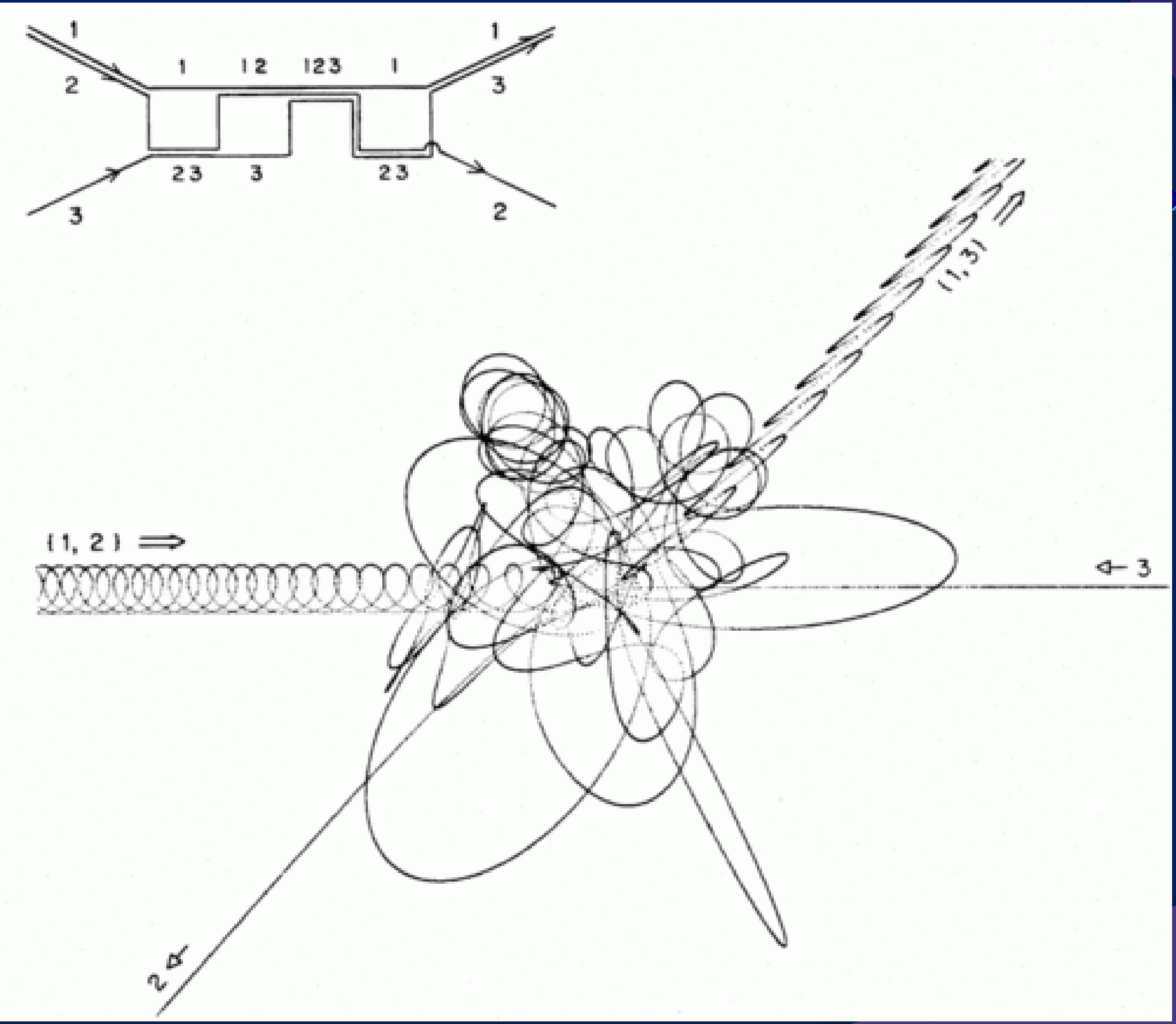
- If the slight change isn't detectable (by measuring instruments) then we won't be able to predict which final state will occur.
- Poincaré's research proved that the problem of determinism and the problem of predictability are distinct problems.

# Chaotic orbits



# Chaotic orbits

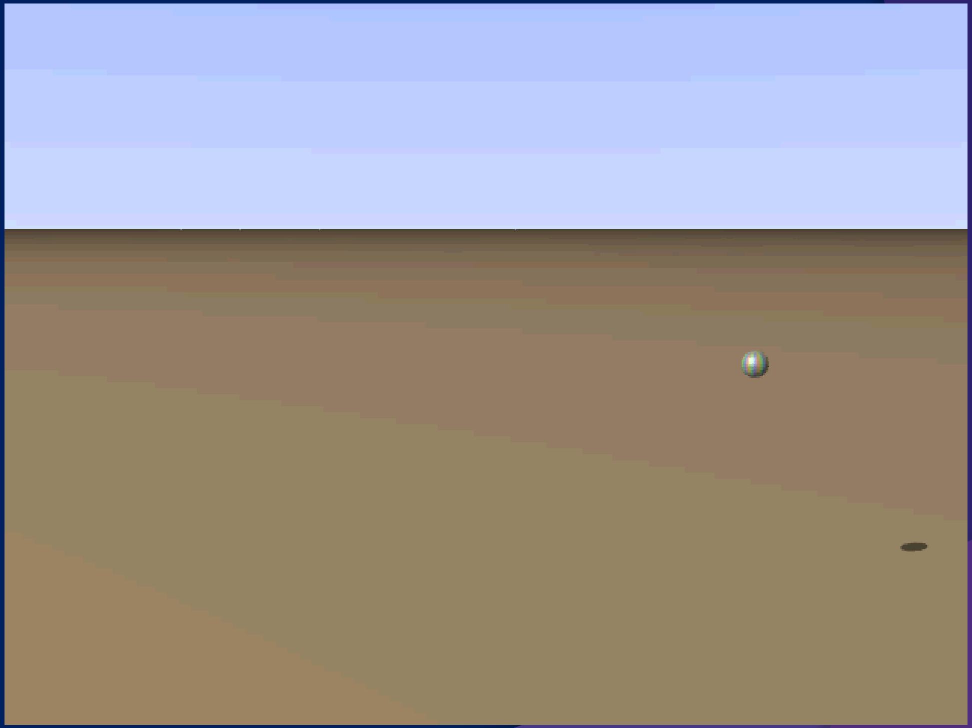





http://users.soe.ucsc.edu/~charlie/3body/

# Poincaré section

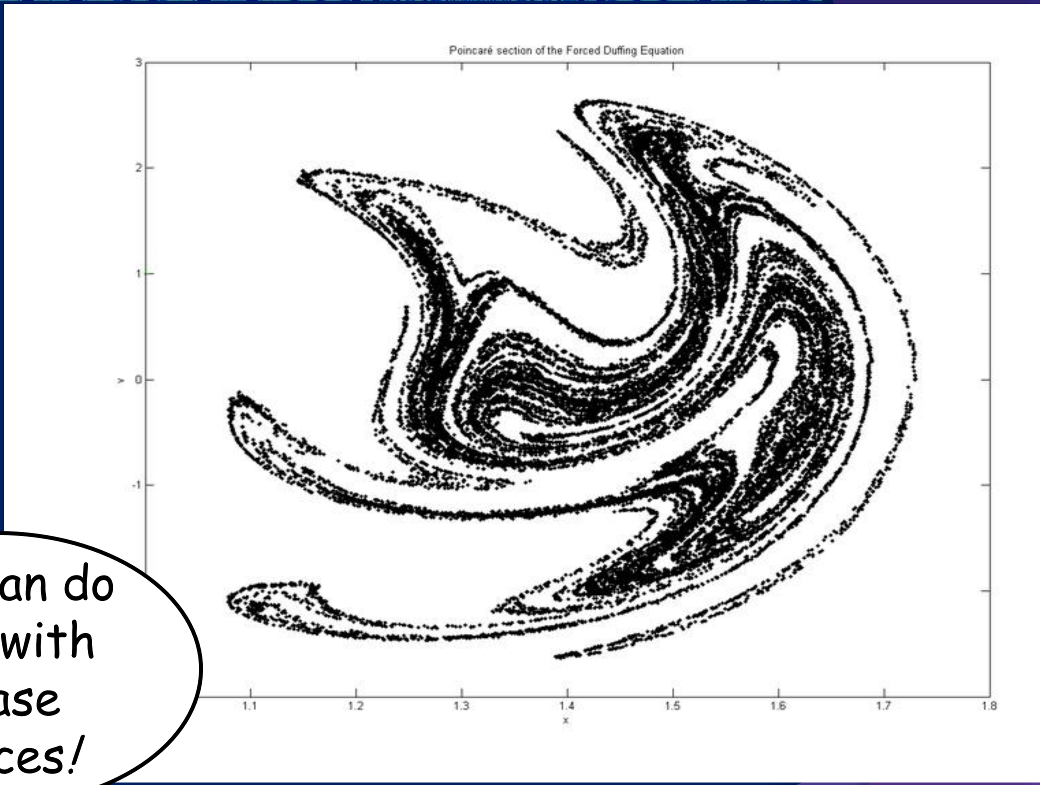
The idea is to cut the orbits with a plan judiciously placed.

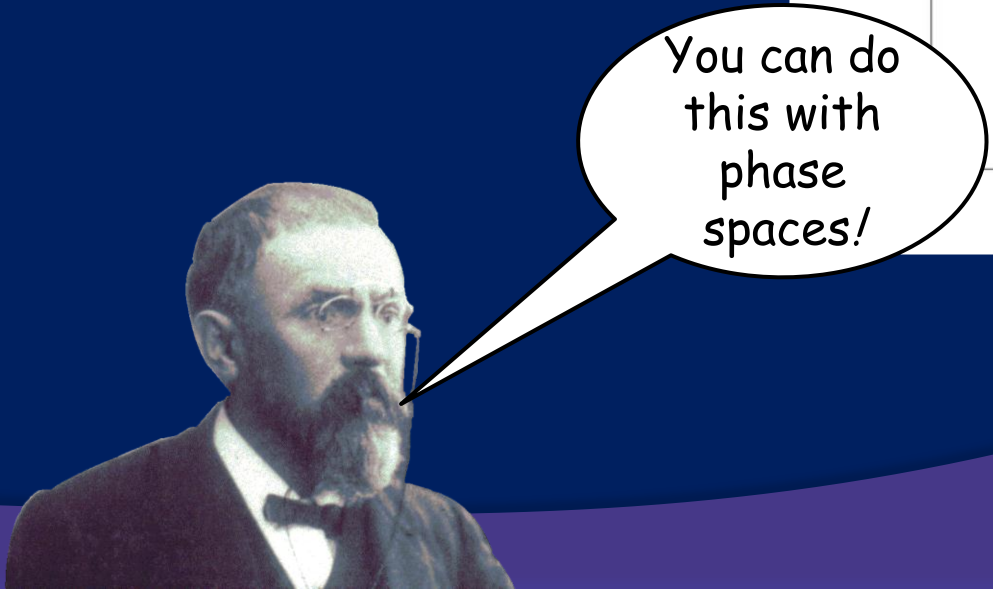


# Poincaré section

## “Poincaré map”

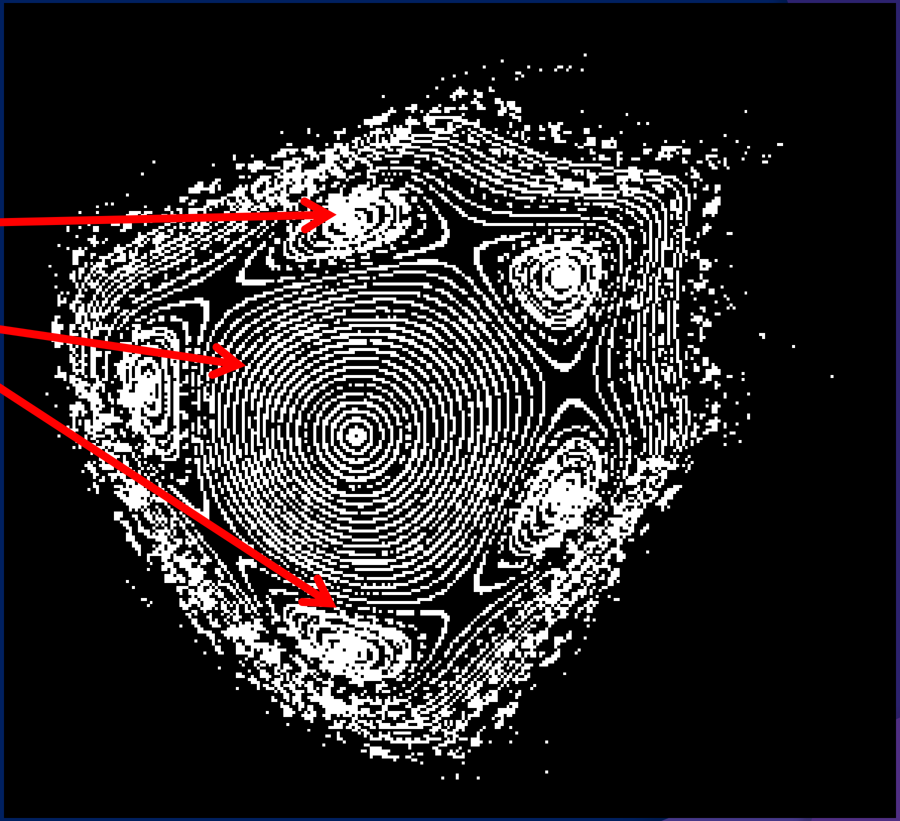
- The Poincaré section reduces a continuous flow to a discrete-time mapping.



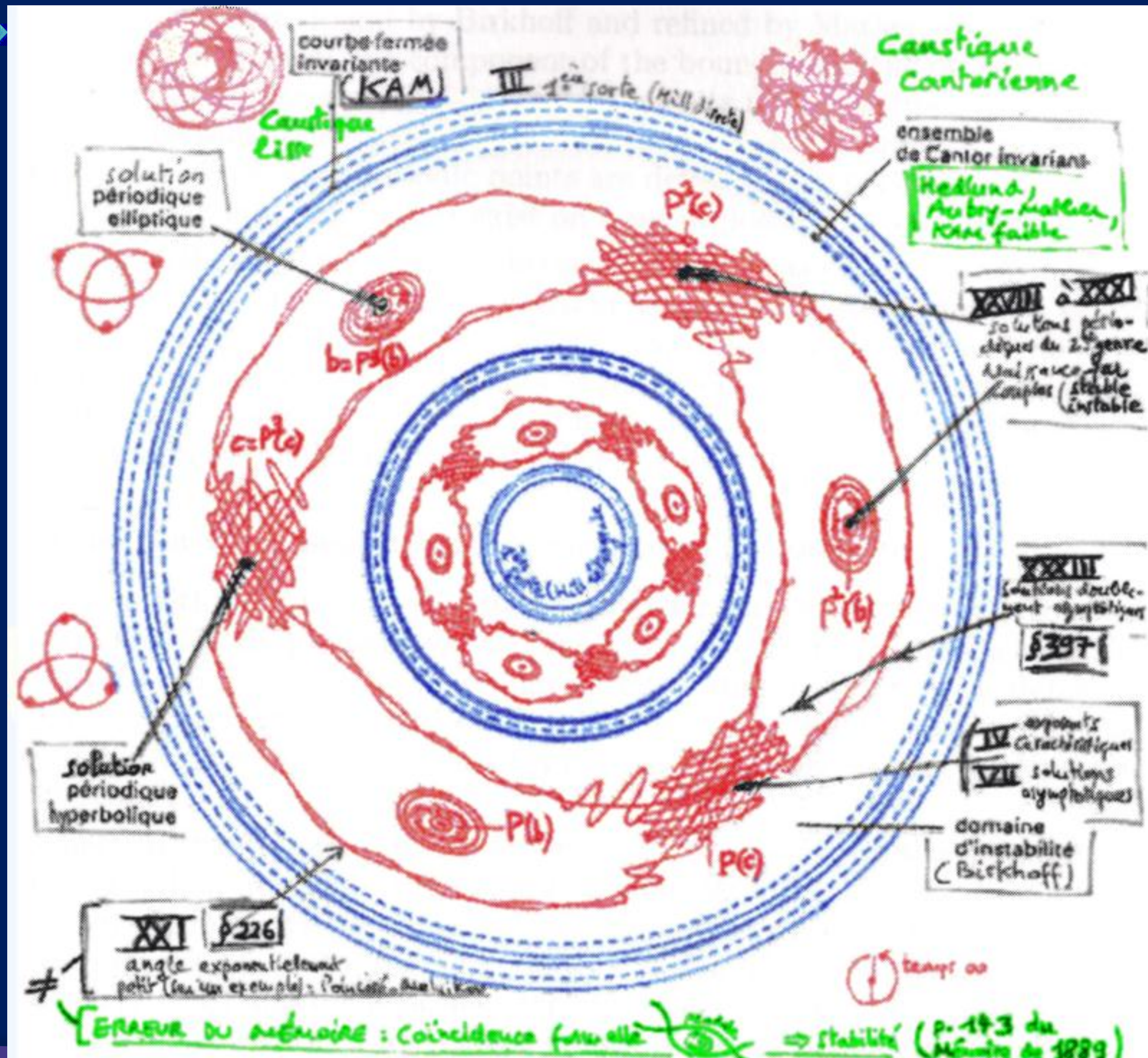
# Singular regions

- Regular orbits
- Resonance regions
- Chaotic zones around the islands (libration).



- The transition from one kind of movement to the other is very sensitive to initial conditions and very unstable.

# Planar circular restricted three-body problem in the case of a high Jacobi constant

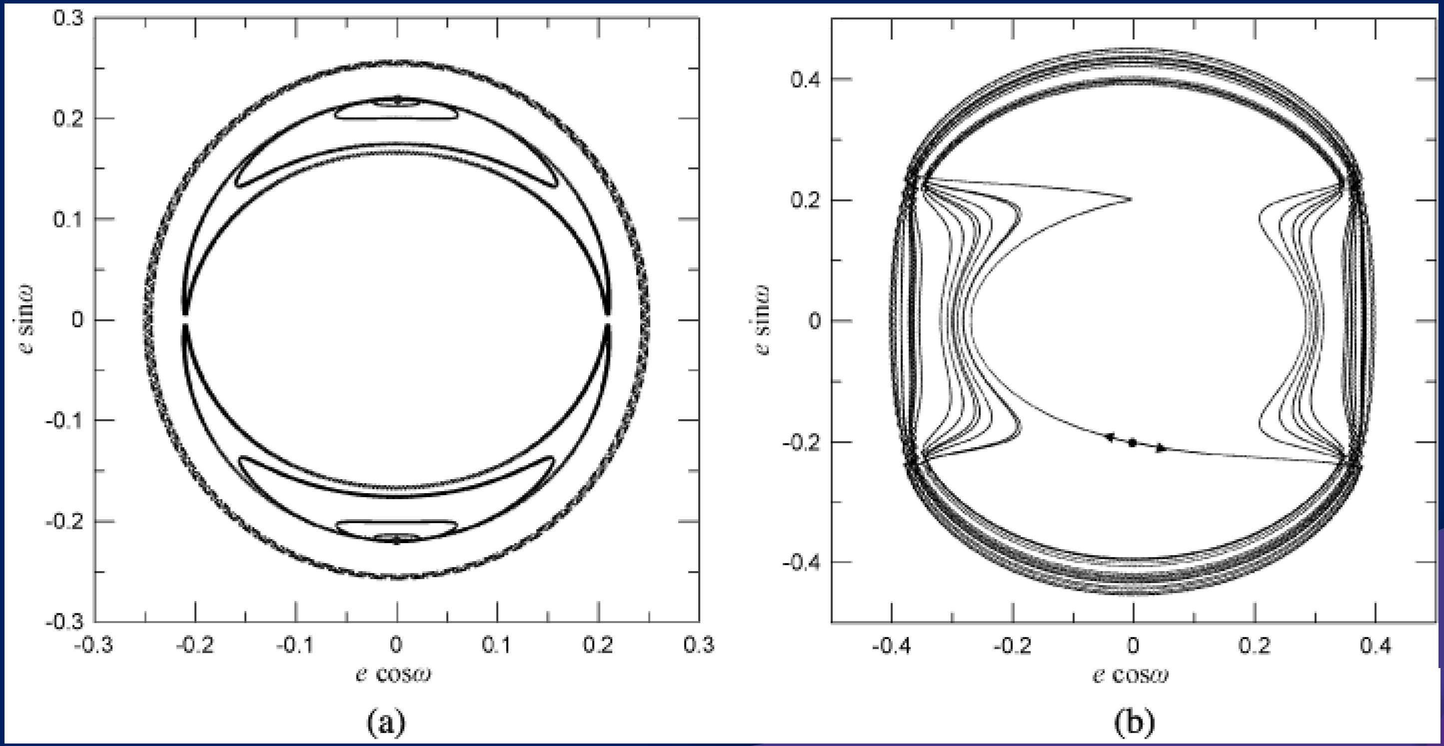



Chenciner (2012)

# Singular regions

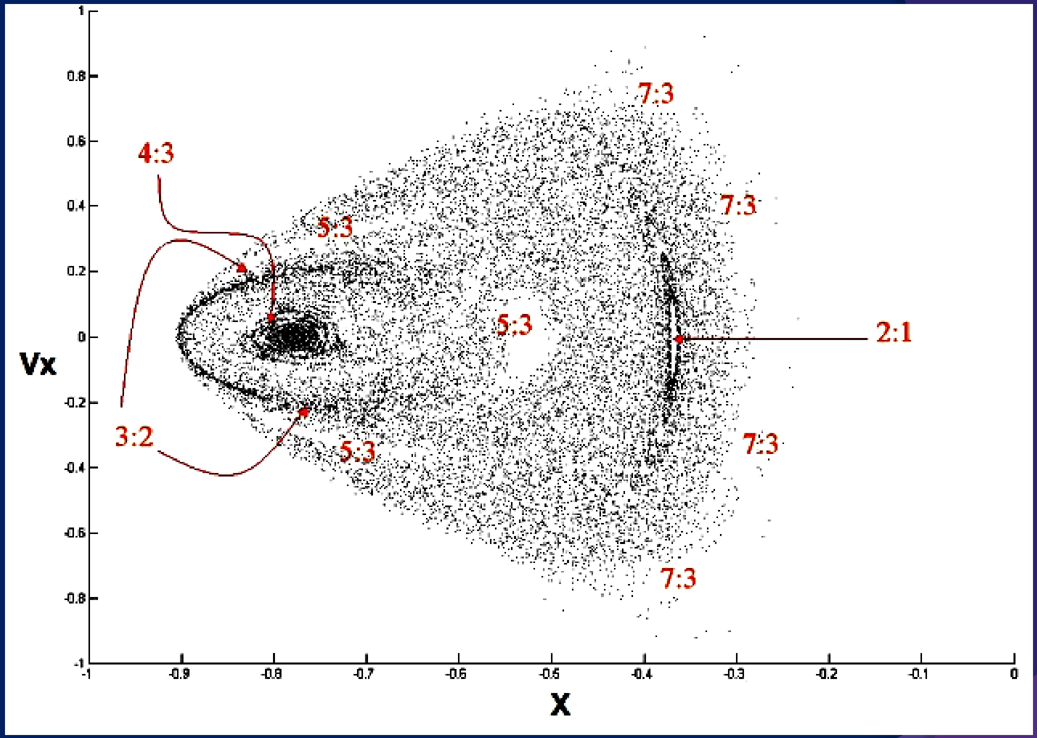
"Three dimensional periodic orbits in exterior mean motion resonances with Neptune"

T. A. Kotoulas & G. Voyatzis, A&A, 2005



# Singular regions

Study of resonances
in the Sun-Jupiter
system.

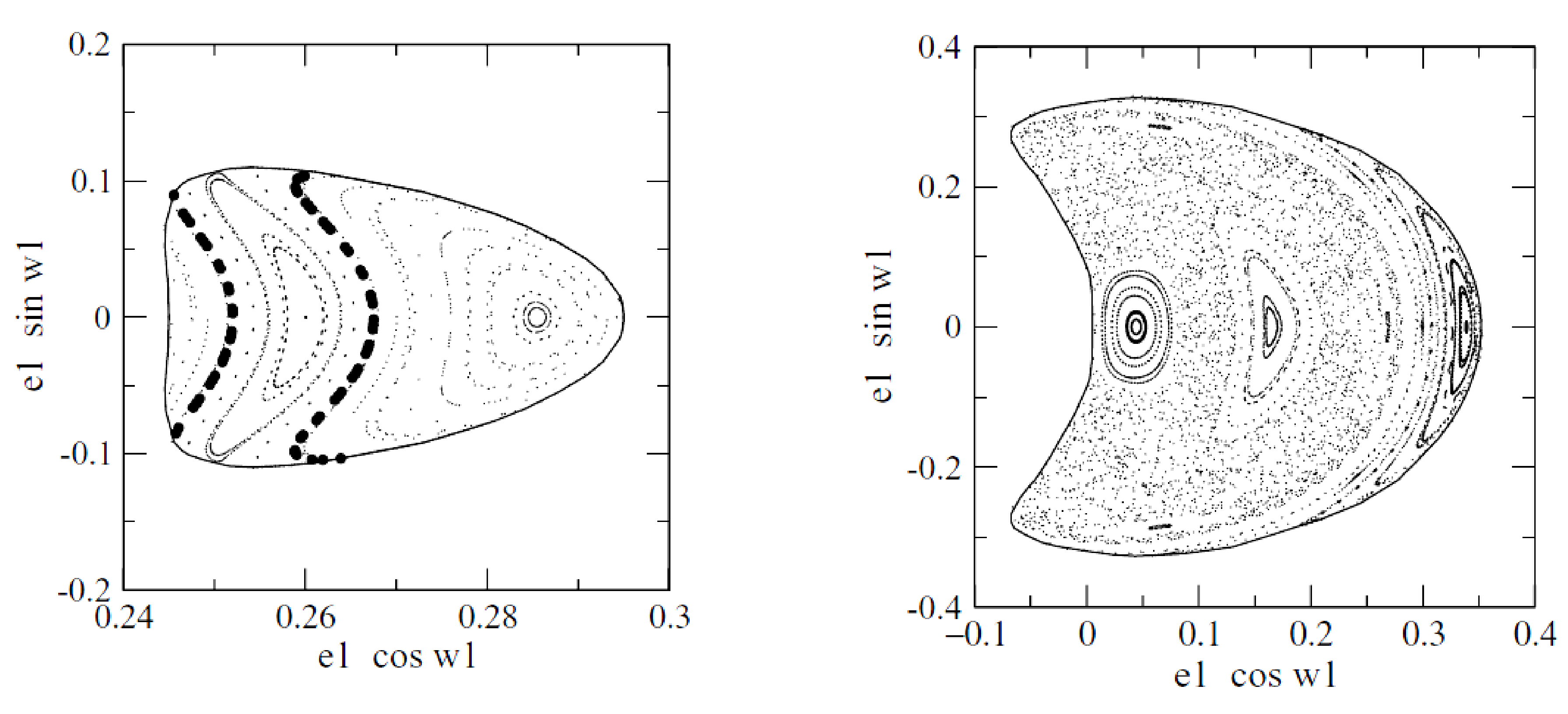



Some resonance value (ratio) of the periodic orbits,
which’s appeared on Poincaré Map.
-Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross (2000)

**Modelling the high-eccentricity planetary three-body problem. Application to the GJ876 planetary system**

C. Beaugé
T. A. Michtchenko

Mon. Not. R. Astron. Soc. **341,** 760–770 (2003)



# 3. The Poincaré Conjecture

# Millennium Problems

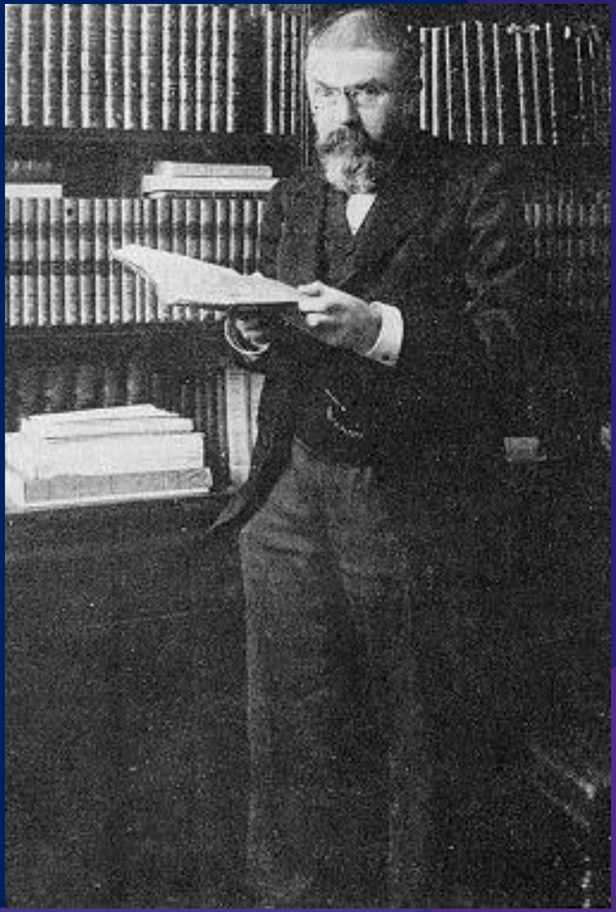
The Clay Mathematics Institute of Cambridge, Massachusetts has named seven mathematical problems (prize: $\$10^6$) http://www.claymath.org/millennium/

1. P versus NP problem
2. Hodge conjecture
3. Poincaré conjecture
4. Riemann hypothesis
5. Yang–Mills existence and mass gap
6. Navier–Stokes existence and smoothness
7. Birch and Swinnerton-Dyer conjecture

# The Poincaré conjecture (1904)

*"Every closed simply connected 3-dimensional manifold is homeomorphic to the 3-dimensional sphere"*

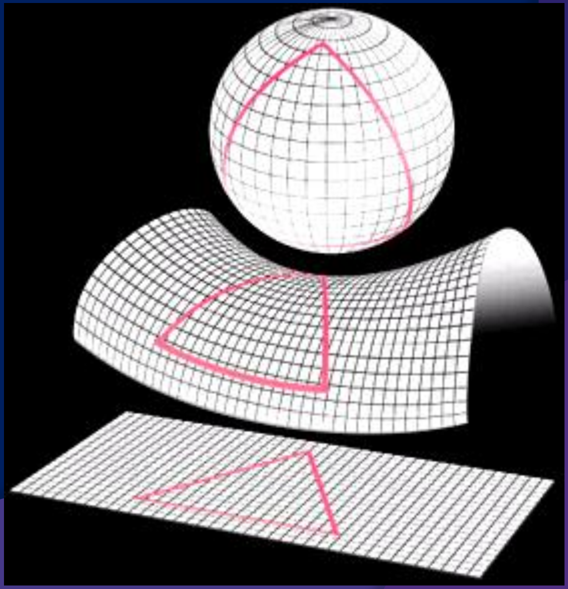
*"Toda variedade compacta tridimensional na qual qualquer caminho fechado possa se contrair a um ponto é homeomorfa à esfera tridimensional."*



# The Poincaré conjecture (1904)

The interpretation of this conjecture is connected to

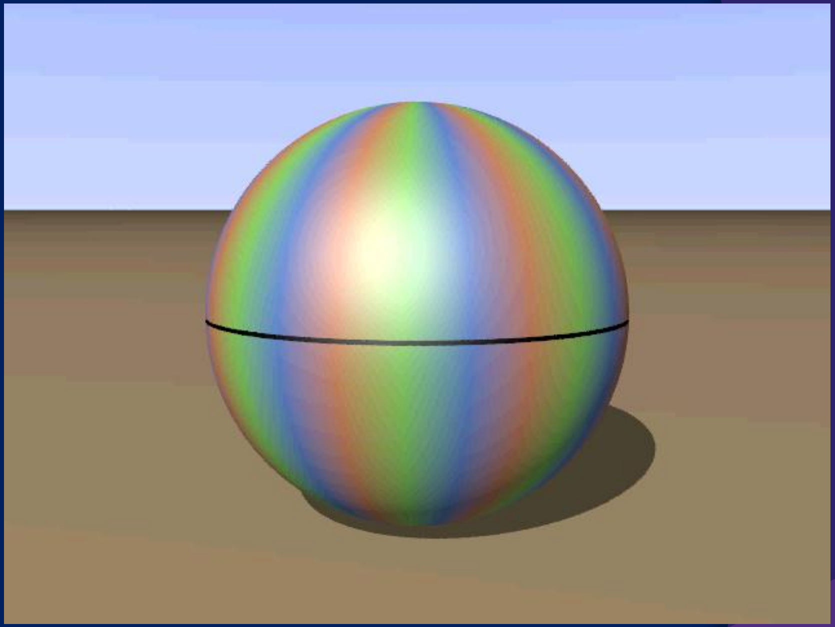
- New directions in mathematics
- Geometry of The Universe



# Closed simply connected

An object in common space is simply connected if it consists of one piece and does not have any "holes" that pass all the way through it.

It can be tested with a loop in the surface that can be contracted to a point.



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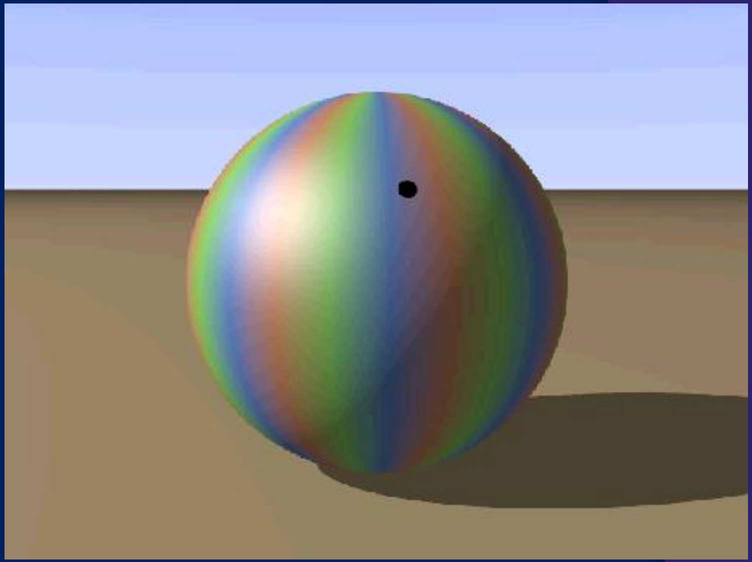
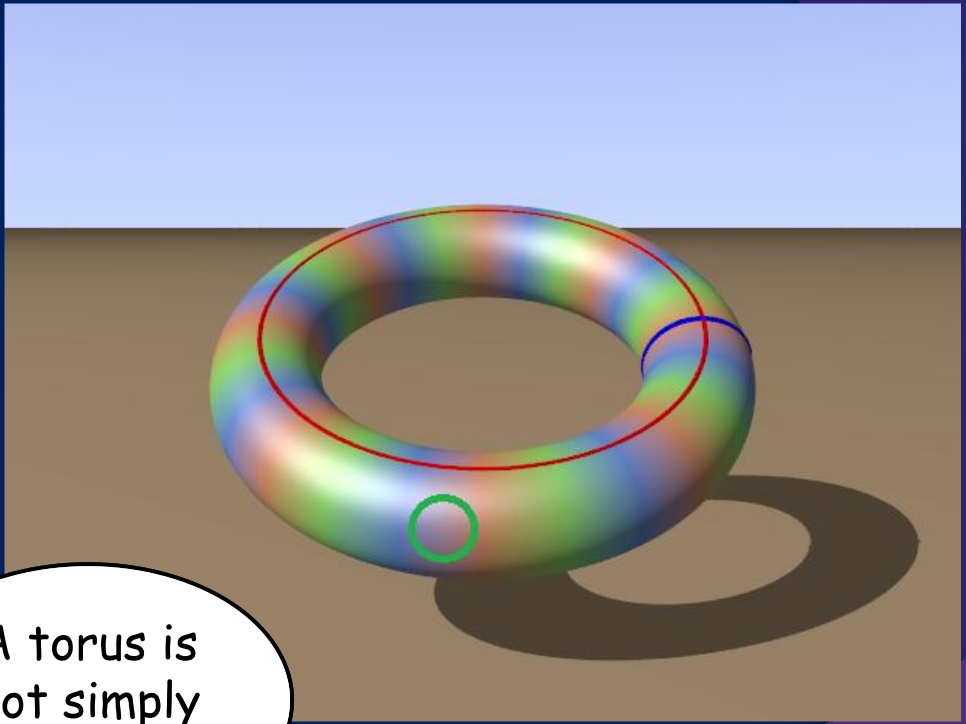
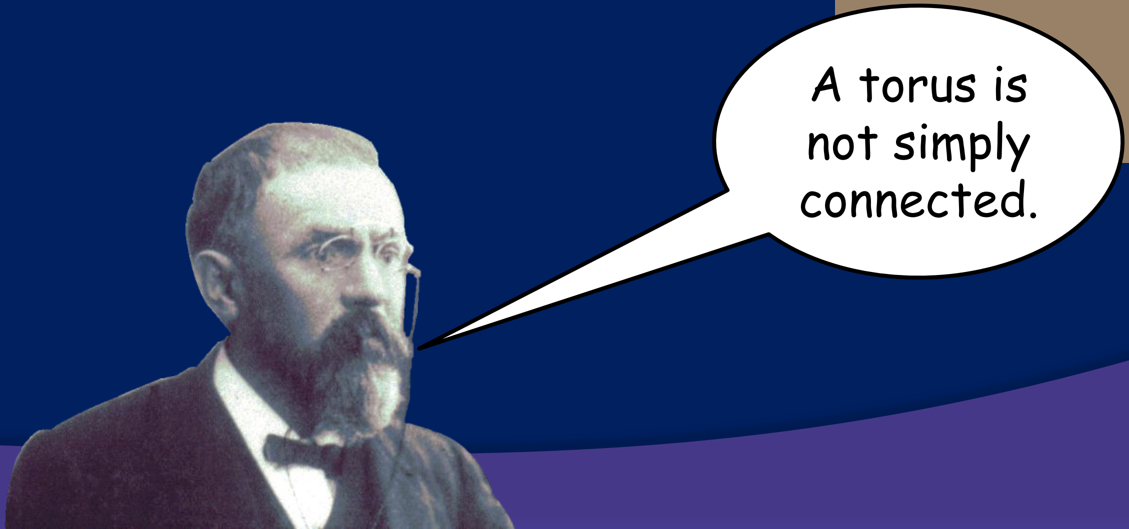


# A torus

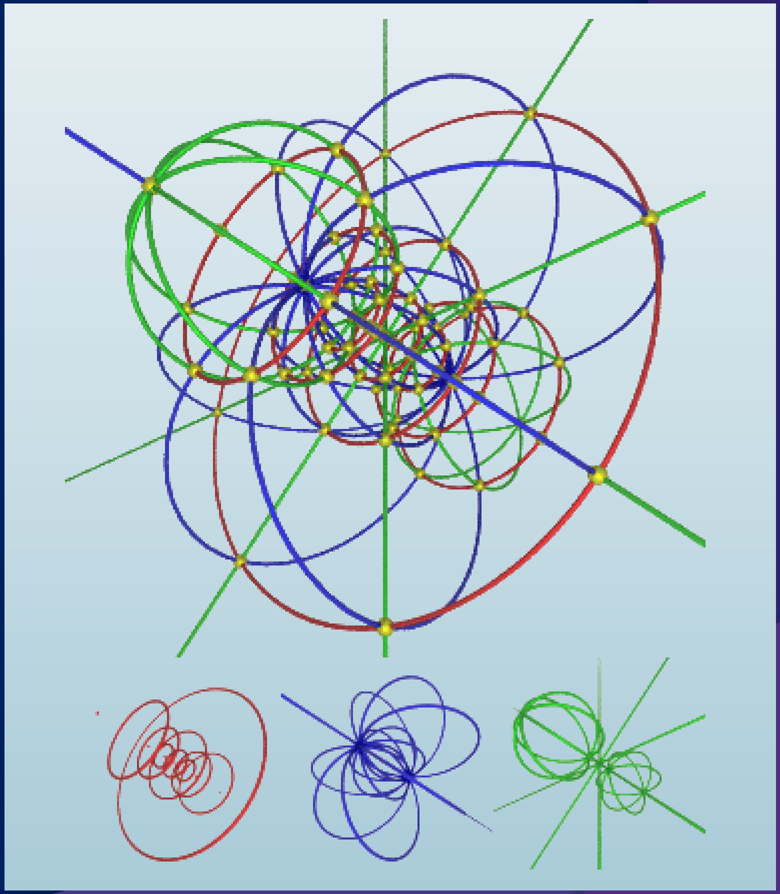
Some loops in the surface of a torus cannot be contracted to a point.





# 3-dim sphere

- The set of points in 4-dim space on the same distance from a given point.

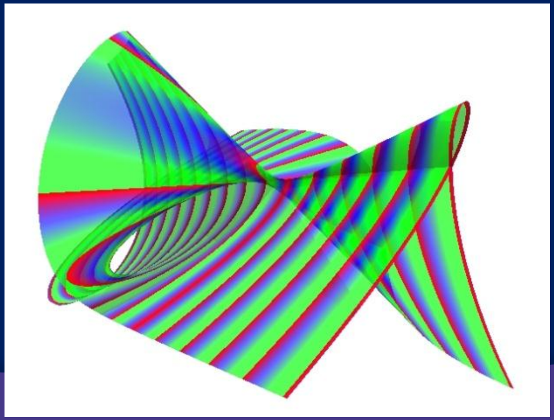


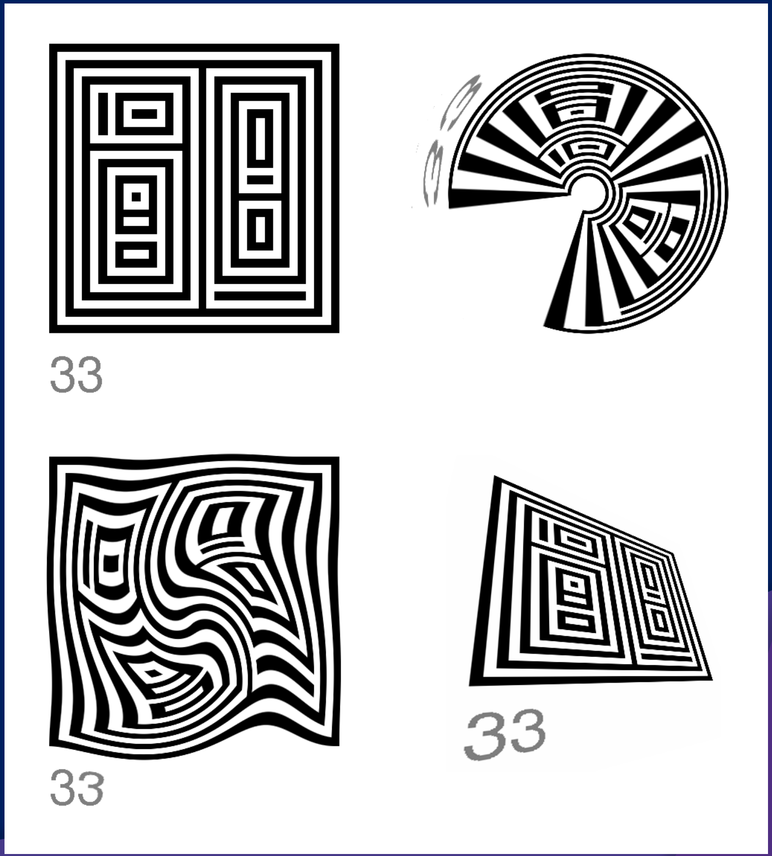
The set of points in 4-space ($x_1$, $x_2$, $x_3$, $x_4$) that define an 3-sphere, (**S**$^3$) is represented by

$$r^2 = \sum_{i=1}^{4} \left(x_i - c_i\right)^2$$

# Topological transformations

“inverting” a torus

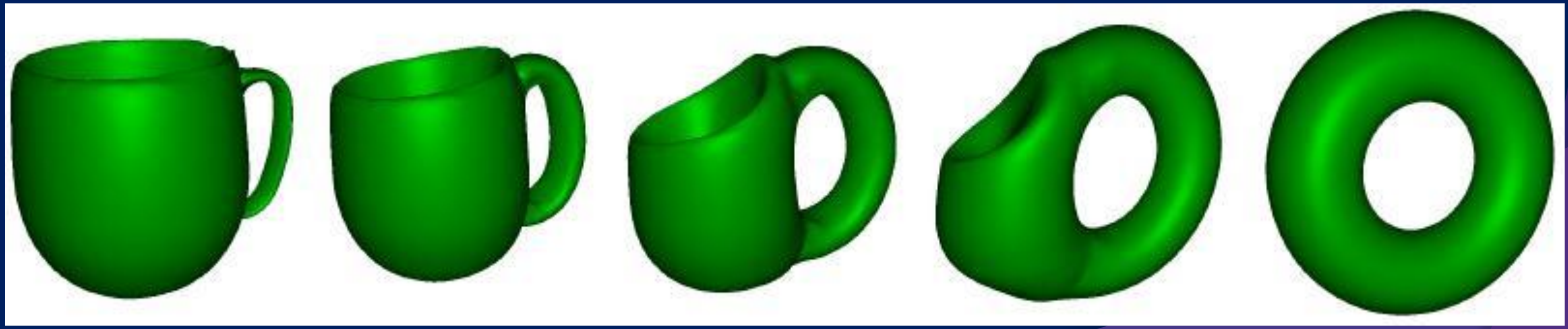




# Topological varieties

"Manifolds"

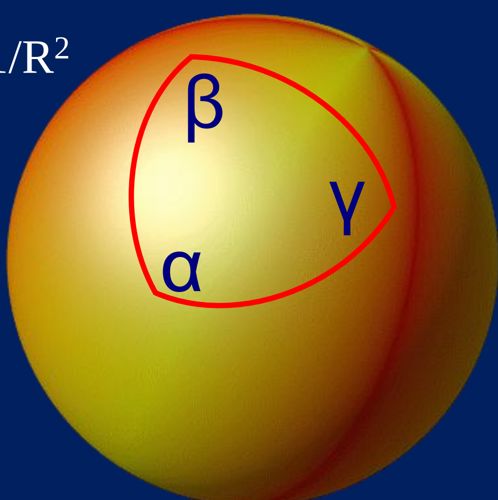
All shapes in all dimensions can be reduced to spheres or *n*-holes torus via topological transformations.



# Manifolds with constant curvature

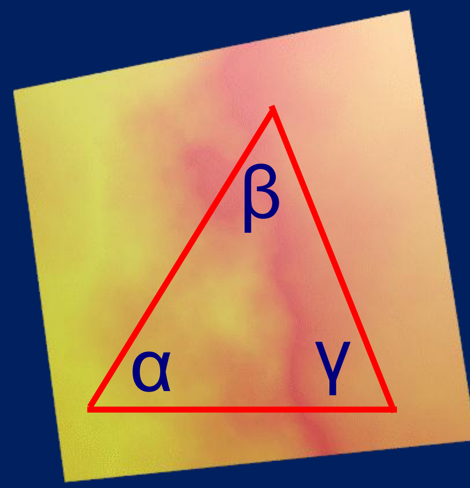
Sphere K>0

$K = 1/R^2$



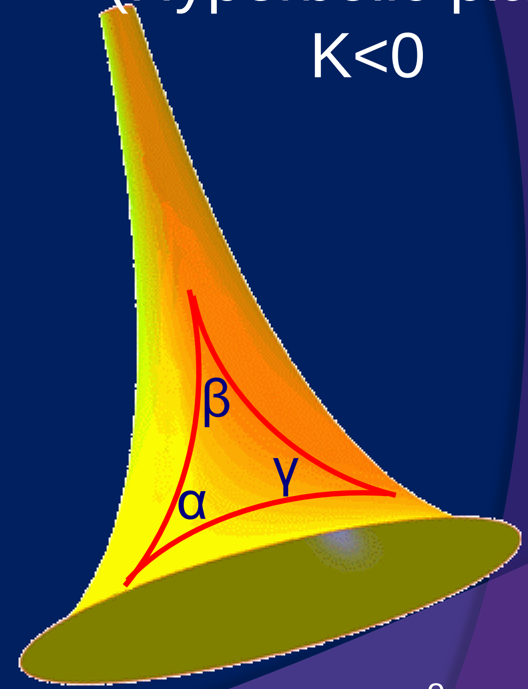

$\alpha + \beta + \gamma > 180^o$

Plane K =0




$\alpha + \beta + \gamma = 180^o$

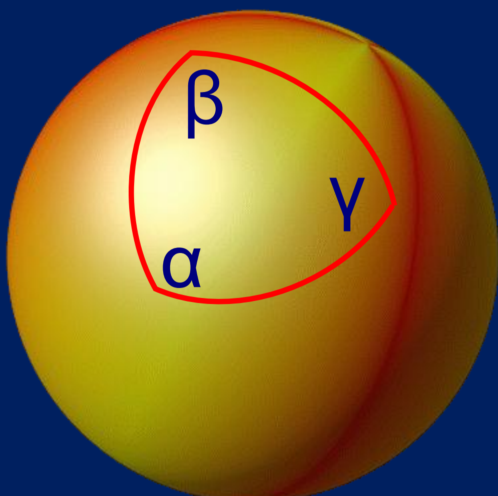
Pseudosphere (Hyperbolic plane) K<0




$\alpha + \beta + \gamma < 180^o$

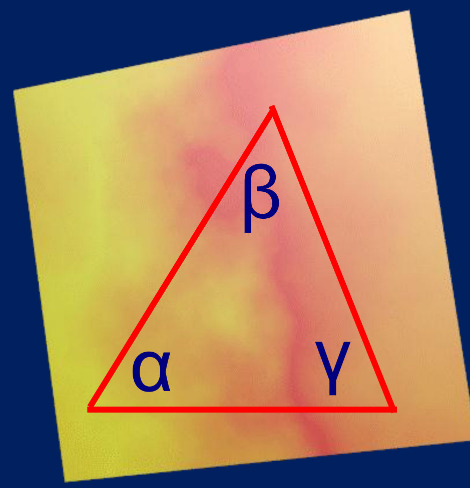
# Models of the Universe

Elliptic
K<0



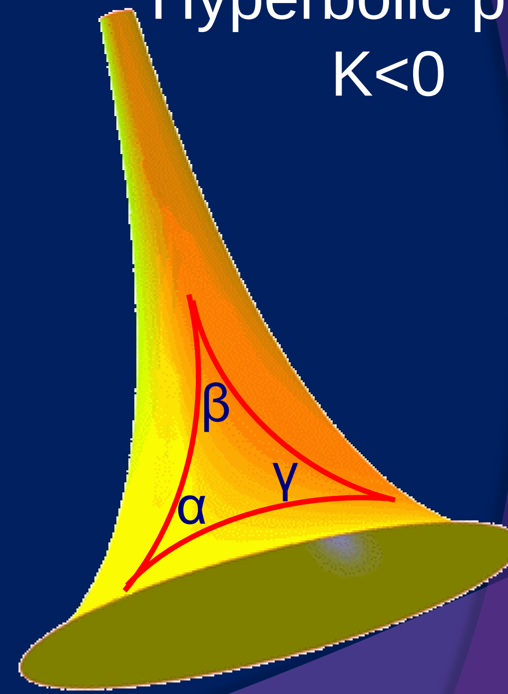

(closed)

Euclidean
K=0




(flat)

Hyperbolic plane
K<0

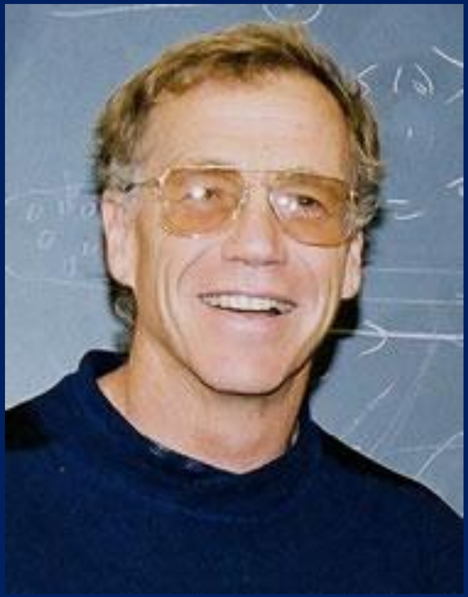



(open)

# Stephen Smale

- Confirmed Poincaré Conjecture for dimension $n \geq 7$ (1960).
- Extended the proof to $n \geq 5$ (with Stallings & Zeeman) Fields Medal in 1966.
- Actually Professor at the Toyota Technological Institute at Chicago.

# Michael H. Freedman

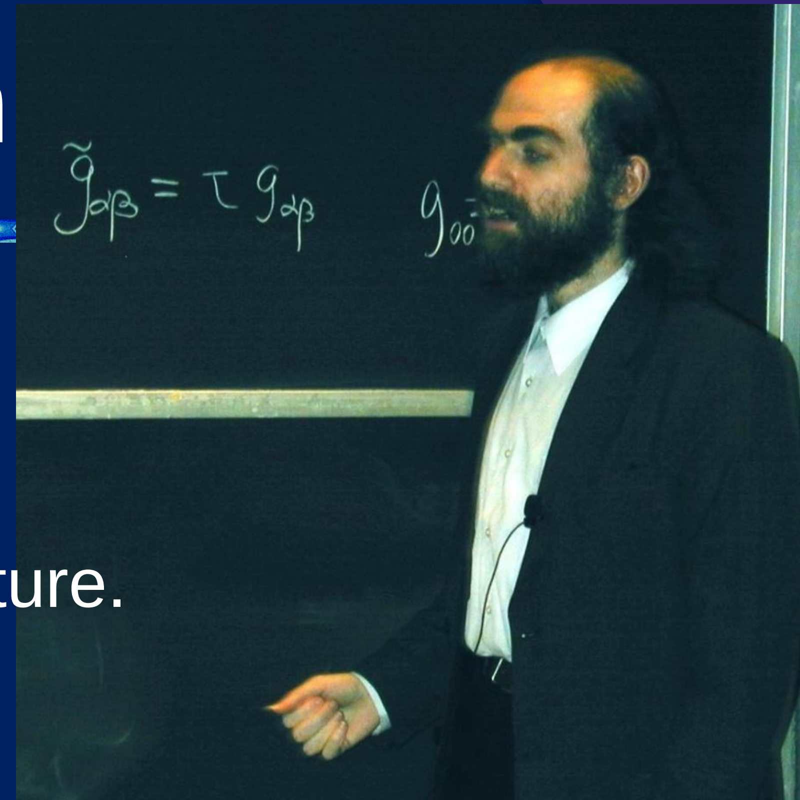


- Confirmed Poincaré conjecture for dimension *n* = 4 (1982);
- Fields Medal in 1966.

- Actually working on topological quantum computer theory.

http://research.microsoft.com/en-us/labs/stationq/

# Grigori I. Perelman



- In 2002 and 2003 posted to the preprint server arXiv.org three papers.
- Confirmed Poincaré conjecture.
- Did not publish the proof in any journal.
- On August 2006, was awarded the Fields Medal at the International Congress of Mathematicians in Madrid.
- Also, the Clay Institute awarded him.
- Perelman declined to accept both awards.

# Xiping & Huaidong

- In June 2006, Zhu Xiping and Cao Huaidong published the 328 pages paper *"A Complete Proof of the Poincaré and Geometrization Conjectures - Application of the Hamilton-Perelman Theory of the Ricci Flow"* in the Asian Journal of Mathematics

# ... now we know


Does every closed simply connected 3-dimensional manifold is homeomorphic to the 3-dimensional sphere?
Yes.

# 4. A bit of Poincaré's geometry

# The new non-Euclidean Poincaré view of the world

- Poincaré obtained many results that are critical to the qualitative theory of differential equations.
- Examples:
  - Poincaré sphere
  - Poincaré map

# Poincaré half-plane model

Metric: $\{\langle x, y\rangle \mid y > 0\}$

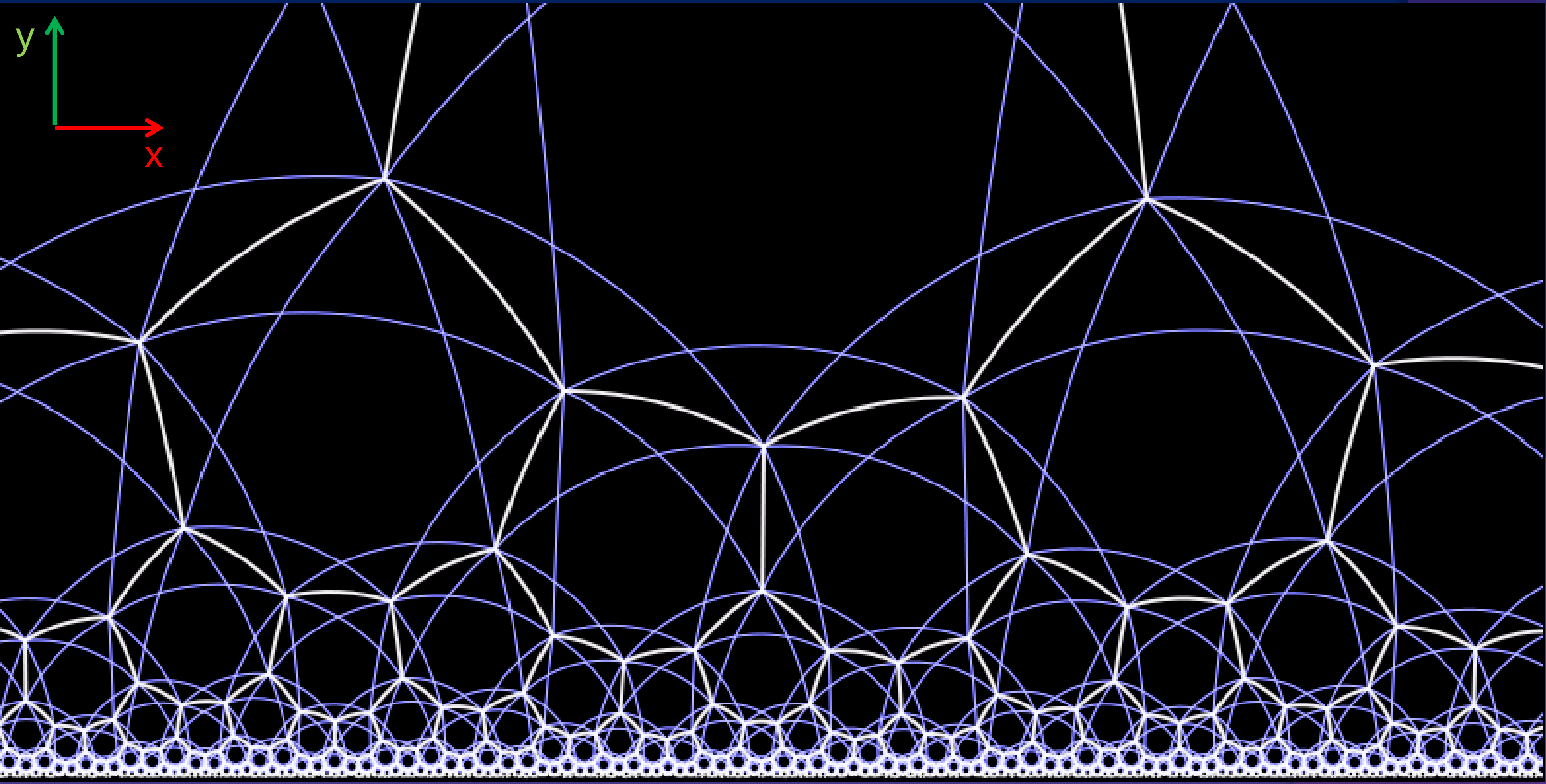
The metric of the model on the half-plane is given by $(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}$

where $s$ measures length along a possibly curved line. The straight lines in the hyperbolic plane (geodesics for this metric tensor, i.e. curves which minimize the distance) are represented in this model by circular arcs perpendicular to the $x$-axis (half-circles whose origin is on the x-axis) and straight vertical lines ending on the $x$-axis. The distance between two points measured in this metric along such a geodesic is

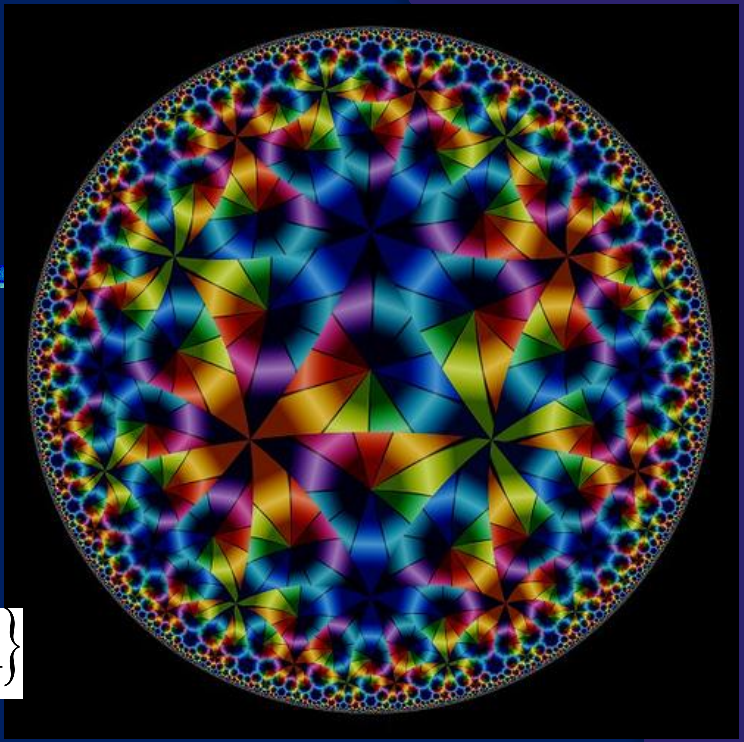
$$\operatorname{dist}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = \operatorname{arc}\cosh\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1y_2}\right)$$

This model is conformal: the angles measured at a point are the same in the model as they are in the actual hyperbolic plane.

# Poincaré half-plane model



# Poincaré disk model



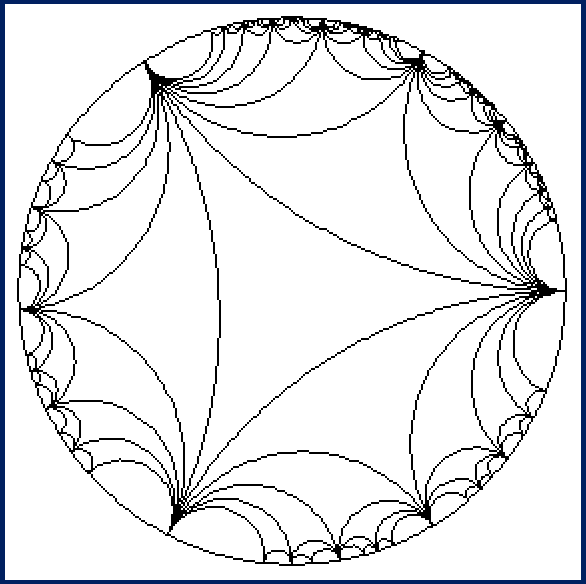
The Poincaré Hyperbolic Disk is a two-dimensional space having hyperbolic geometry defined as the disk $\{x \in \Re^2 : |x| < 1\}$ with hyperbolic metric

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{\left(1 - x^2 - y^2\right)^2}$$

- The Poincaré hyperbolic disk represents a conformal map, so angles between rays can be measured directly.

- There is an isomorphism between the Poincaré disk model and the Klein-Beltrami model.

# Poincaré disk model



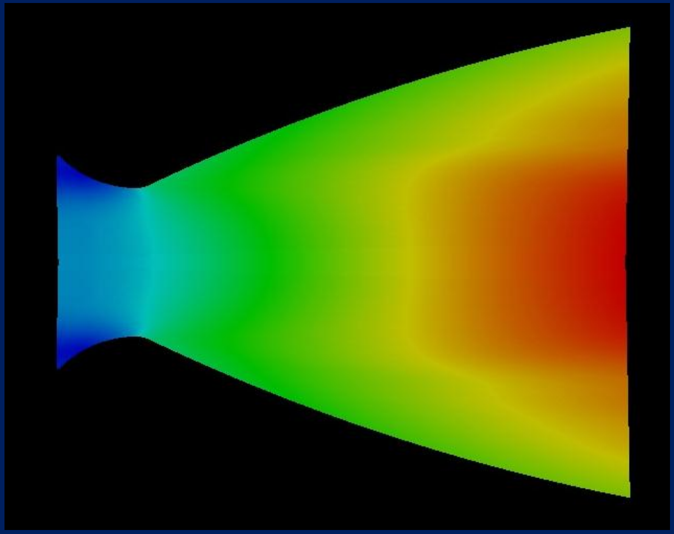
1. The geodesics are straight lines passing through the center of the disc and circular arcs that meet at right angles via edge.
2. Angles between intersecting geodesics coincide with the Euclidean angles.
3. For every point and in any direction, there is exactly one geodesic.

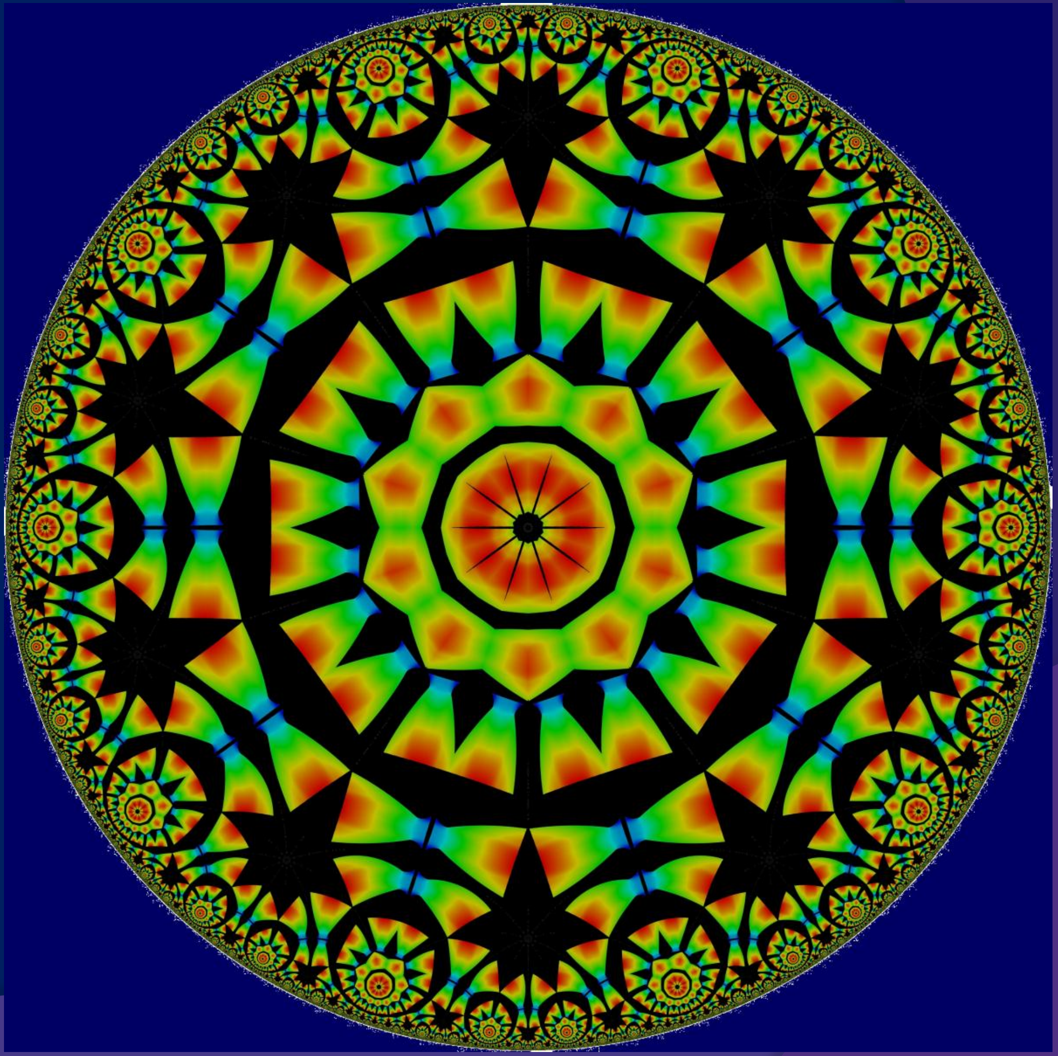
# Escher's Circle Limits

*"Portrait"*
M.C. Escher

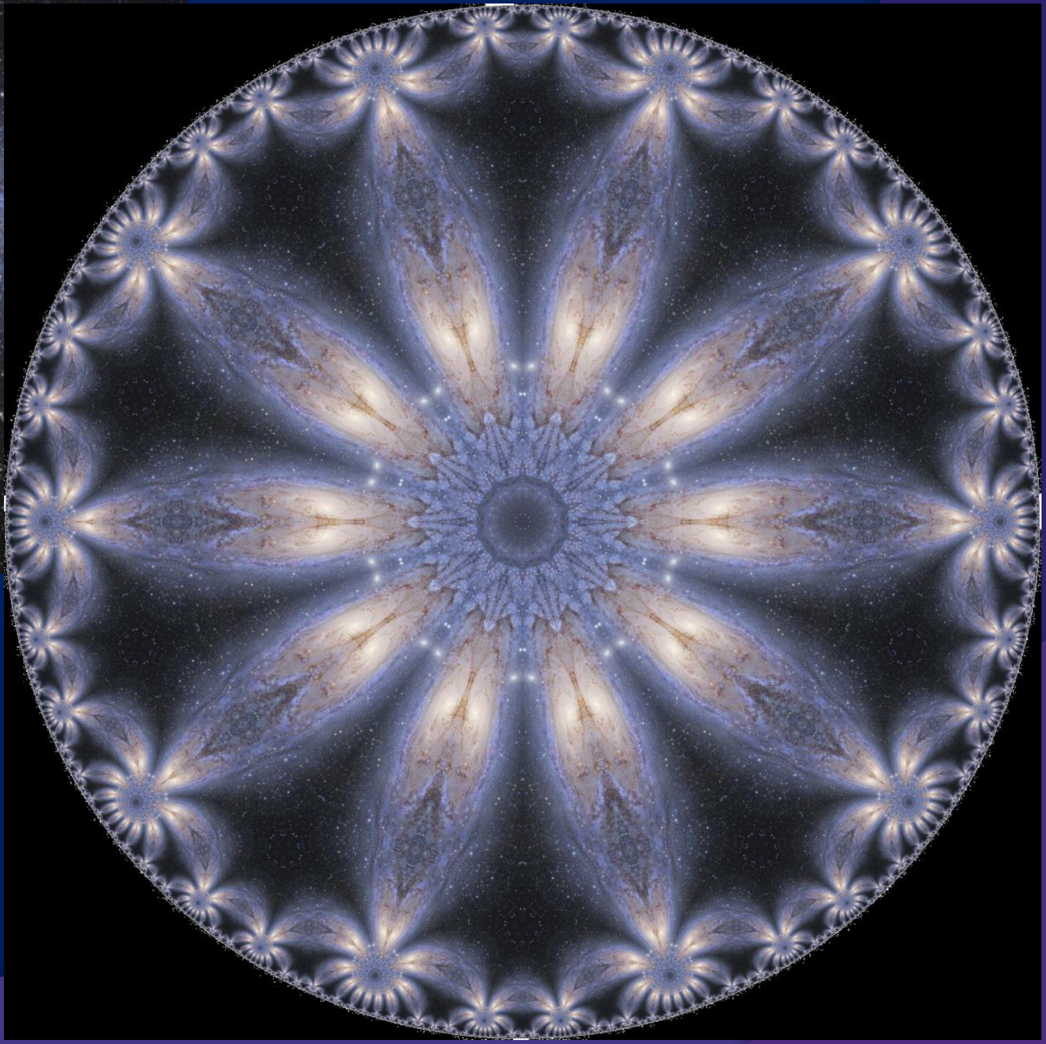
- Tilings of the Poincaré Disk are of interest to both mathematicians and artists.
- M.C. Escher created four wood carvings based on tilings of the Poincaré disk titled.
- These images are remarkable for both their aesthetic beauty as well as the mathematical skill Escher acquired to create them.

# Homemade tilings

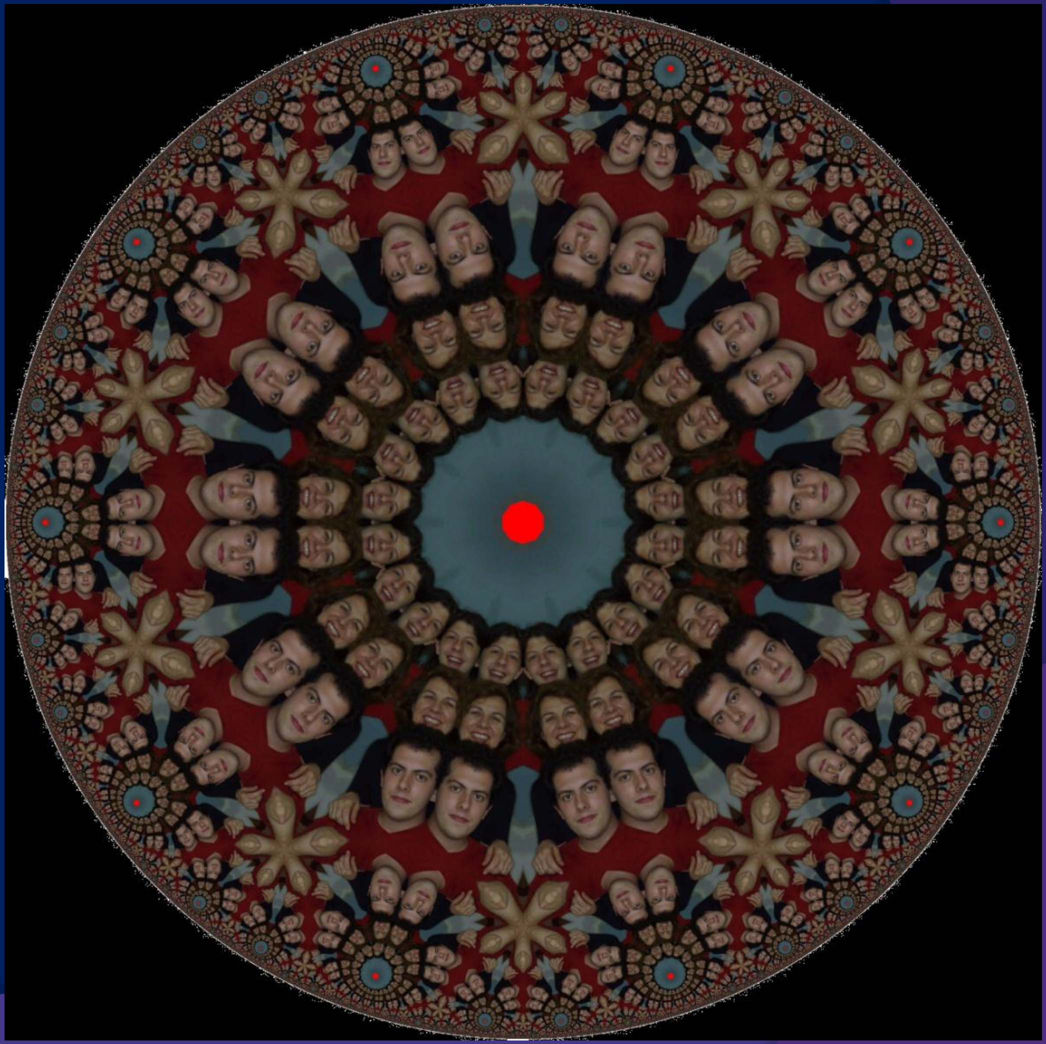




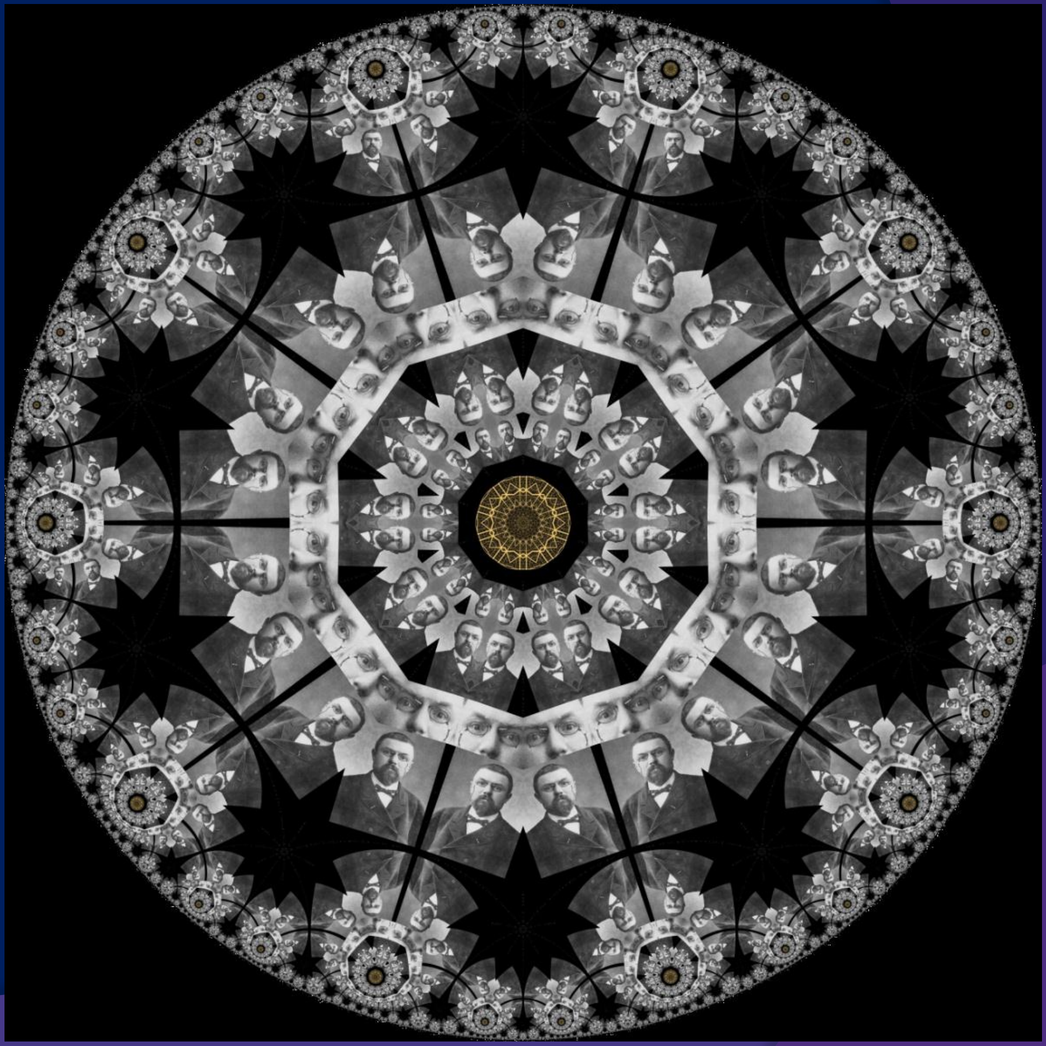
# Homemade tilings



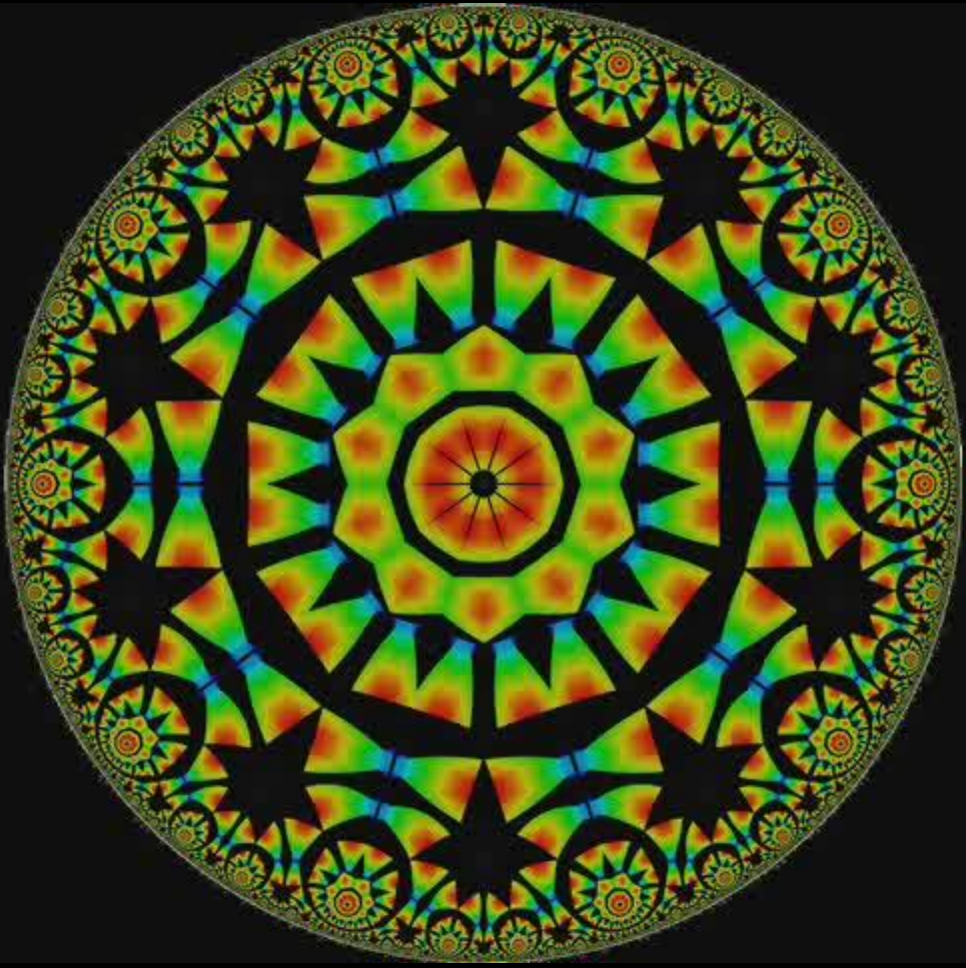
# Homemade tilings



# Homemade tilings




C'est assez.
Je vous remercie de
votre attention!



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